Total No. of Printed Pages—23

**5 SEM TDC DSE MTH (CBCS) 2.1/2.2/2.3/2.4 (H)**

# 2022

( Nov/Dec )

## MATHEMATICS

( Discipline Specific Elective )

( For Honours )

Paper : DSE–2.1/2.2/2.3/2.4

*The figures in the margin indicate full marks for the questions*

Paper : DSE–2.1

**( Mathematical Modelling )**

*Full Marks* : 60

*Pass Marks* : 24

*Time* : 3 hours

**1.** *(a)* What do you mean by a power series solution to a differential equation? 1

*(b)* What is regular point of a differential equation? 1

2. *(a)* Find the regular-singular points of the differential equation

$$(1-x^2)y''-2xy'+\lambda(\lambda+1)y=0$$

where $\lambda$ is a real constant. 3

*(b)* Solve the following Bessel equation of order zero : 6

$$x^2y''+xy'+x^2y=0$$

*Or*

Consider the Legendre's equation

$$(1-x^2)y''-2xy'+\lambda(\lambda+1)y=0,$$

$\lambda$ is non-negative. Show that if $\lambda$ is a positive odd integer, $2n+1$, then the series solution $y_2$ reduces to a polynomial of degree $2n+1$ containing only odd powers of $x$.

3. *(a)* Find the inverse Laplace transform of $\left(\frac{1}{1+s}\right)$ and $\left(\frac{1}{s-1}\right)$. 2

*(b)* State and prove first shifting theorem of Laplace transform. 3

*(c)* Find the inverse Laplace transform of

$$\frac{30}{s^7}+\frac{8}{s-4}$$ 4

(d) Solve the initial-value problem using the Laplace transform $y'' + 9y = 27t^3$ with $y(0) = 0$ and $y'(0) = 0$. 5

4. (a) Write the importance of Monte Carlo simulation technique. 2

(b) Explain classical Monte Carlo method with an example. 3

5. (a) Write two disadvantages of middle square method for generation of random number. Write the algorithm for generation of random number using middle square method. 5

(b) What is linear congruence method? How does it work? Explain with an example. 5

6. (a) Explain about morning rush hour queuing model with an example. 5

*Or*

(b) Explain about Harbor model with an example.

**7.** Answer any *three* of the following questions : 5×3=15

*(a)* A nutritionist advises an individual who is suffering from iron and vitamin B deficiency to take at least 2400 mg (milligrams) of iron, 2100 mg of vitamin B1 (thiamine), and 1500 mg of vitamin B2 (riboflavin) over a period of time. Two vitamin pills are suitable, brand–*A* and brand–*B*. Each brand–*A* pill costs ₹ 60 and contains 40 mg of iron, 10 mg of vitamin B1 and 5 mg of vitamin B2. Each brand–*B* pill costs ₹ 80 and contains 10 mg of iron and 15 mg each of vitamins B1 and B2. What combination of pills should the individual purchase in order to meet the minimum iron and vitamin requirements at the lowest cost? (Use graphical method) :

| | *Brand–A* | *Brand–B* | *Minimum requirement* |
|---|---|---|---|
| Iron | 40 mg | 10 mg | 2400 mg |
| Vitamin B1 | 10 mg | 15 mg | 2100 mg |
| Vitamin B2 | 5 mg | 15 mg | 1500 mg |
| Cost/Pill | ₹ 60 | ₹ 80 | |

(b) A manufacturer produces three types of plastic fixtures. The time required for molding, trimming and packaging is given in the following table (Times are given in hours per dozen fixtures). How many dozens of each type of fixture should be produced to obtain a maximum profit?

| | *Type-A* | *Type-B* | *Type-C* | *Total time available* |
|---|---|---|---|---|
| Molding | 1 | 2 | 3/2 | 1200 |
| Trimming | 2/3 | 2/3 | 1 | 4600 |
| Packaging | 1/2 | 1/3 | 1/2 | 2400 |
| Profit | ₹ 1,100 | ₹ 1,600 | ₹ 1,500 | — |

(c) A company has two grades of inspectors, *I* and *II* to undertake quality control inspection. At least 1500 pieces must be inspected in an 8-hour day. Grade *I* inspector can check 20 pieces in an hour with an accuracy of 96%. Grade *II* inspector checks 14 pieces an hour with an accuracy of 92%. Wages of grade *I* inspector are ₹ 5 per hour while those of grade *II* inspector are ₹ 4 per hour. Any error made by an inspector costs ₹ 3 to the company.

If there are, in all, 10 grade *I* inspectors and 15 grade *II* inspectors in the company, find the optimal assignment of inspectors that minimize the daily inspection cost.

*(d)* A manufacturer of cylindrical containers receives tin sheets in widths of 30 cm and 60 cm respectively. For these containers the sheets are to be cut to three different widths of 15 cm, 21 cm and 27 cm respectively. The number of containers to be manufactured from these three widths are 400, 200 and 300 respectively. The bottom plates and top covers of the containers are purchased directly from the market. There is no limit on the lengths of standard tin sheets. Formulate the LPP for the production schedule that minimizes the trim losses.

Paper : DSE–2.2

**( Mechanics )**

*Full Marks* : 80

*Pass Marks* : 32

*Time* : 3 hours

UNIT—1

**1.** *(a)* A force $\vec{F_1} = (10\hat{i} + 6\hat{j} + 3\hat{k})$ acts at position (3, 0, 2). At point (0, 2, −3), an equal but opposite force $-\vec{F_1}$ acts. Calculate the couple moment. 2

*(b)* Show that any number of coplanar couples, acting on a body is equivalent to a single couple whose moment is equal to the algebraic sum of the moments of the couples. 6

*Or*

Show that two couples in the same plane whose moments are equal and of the same sign are equivalent to one another.

**2.** *(a)* Write down the condition of equilibrium of any number of concurrent forces. 2

(b) Three forces $\vec{P}$, $\vec{Q}$, $\vec{R}$ acting along $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are in equilibrium. If $O$ be the circum-centre of the triangle $ABC$, then show that

$$\frac{P}{\frac{1}{b^2}+\frac{1}{c^2}-\frac{a^2}{b^2c^2}}=\frac{Q}{\frac{1}{c^2}+\frac{1}{a^2}-\frac{b^2}{c^2a^2}}=\frac{R}{\frac{1}{a^2}+\frac{1}{b^2}-\frac{c^2}{a^2b^2}}$$

where $a$, $b$, $c$ are the lengths of the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$. 7

*Or*

An electric light fixture weighing 15 N hangs from a point $C$ by two strings $AC$ and $BC$. The string $AC$ is inclined at 60° to the horizontal and $BC$ at 45° to the horizontal. Draw the free body diagram and determine the forces in the strings $AC$ and $BC$.

(c) What do you mean by distributed force system? Give an example of it. 2+1=3

UNIT—2

3. (a) What do you mean by coefficient of friction? Write down the dimension of coefficient of friction. 2+1=3

*(b)* The horizontal position of the 500 kg rectangular block of concrete is adjusted by the 5° wedge under the action of the force *P*. If the coefficient of friction for both wedge surfaces is 0·30 and the coefficient of friction between block and horizontal surface is 0·60, then determine the least force *P* required to move the block. 7

*Or*

A smooth sphere of weight *W*, rests between a vertical wall and a prism, one of whose faces rests on a horizontal plane, if the coefficient of friction between the horizontal and prism is $\mu$. Show that the least weight of the prism consistent with the equilibrium is $W\left(\dfrac{\tan\alpha}{\mu}-1\right)$, where $\alpha$ is the inclination to the horizon of the face in contact with the sphere.

**4.** *(a)* A plane surface is bounded by *x*-axis, the curve $y^2 = 25x$ and the line $x = 10$. Calculate the moments of the area about *x* and *y* axes. What are the centroidal coordinates? 3+2=5

(b) Define second moment of area and product of area. 2+2=4

(c) Find $I_{xx}$, $I_{yy}$ and $I_{xy}$ for the area bounded by the curves $y = x^2$ and $y^2 = 3x$. 2+2+2=6

*Or*

Define polar moment of area. Show that the polar moment of area of a circular area of radius $r$ is $\frac{\pi r^4}{2}$ at the centre.

2+4=6

UNIT—3

5. (a) Show that a conservative force field is a function of position and gradient of a scalar field. 4

(b) Give two examples of conservative force field. 2

(c) Show that the sum of the potential energy and the kinetic energy for a particle remains constant for all times during the motion of the particle.

A particle is dropped with zero initial velocity down a frictionless chute. What is the magnitude of the velocity if the vertical drop during the motion is $h$ ft?

4+3=7

*Or*

Given the following conservative force field :

$$\vec{F} = (10z + y)\hat{i} + (15yz + x)\hat{j} + \left(10x + \frac{15y^2}{2}\right)\hat{k}$$

Find the force potential.

Also, calculate the work done by $\vec{F}$ on a particle going from $\vec{r}_1 = 10\hat{i} + 2\hat{j} + 3\hat{k}$ to $\vec{r}_2 = -2\hat{i} + 4\hat{j} - 3\hat{k}$. 4+3=7

**6.** *(a)* A reference *xyz* is moving such that the origin *O* has at time *t*, a velocity relative to reference *XYZ* given as

$$\vec{V}_O = 6\hat{i} + 12\hat{j} + 13\hat{k}$$

The *xyz* reference has an angular velocity $\vec{\omega}$ relative to *XYZ* at time *t*, given as

$$\vec{\omega} = 10\hat{i} + 12\hat{j} + 2\hat{k}$$

What is the time rate of change of a directed line segment ρ going from (3, 2, −5) to (−2, 4, 6) in *xyz* relative to *XYZ* ? 4

*(b)* What do you mean by translation and rotation of rigid body? 2+2=4

(c) An airplane moving at 200 ft/sec is undergoing a roll of 2 rad/min. When the plane is horizontal, an antenna is moving out at a speed of 8 ft/sec relative to the plane and is at a position of 10 ft from the centre line of the plane. If we assume that the axis of roll corresponds to the centre line, what is the velocity of the antenna end relative to the ground? 5

*Or*

State and prove Chasles' theorem.

**7.** (a) Establish the relation between acceleration vectors of a particle for two systems of references moving arbitrarily relative to each other. 5

*Or*

Find the kinetic energy of rigid body rotating about a fixed point.

(b) Derive the moment of momentum equation for a single particle. 4

Paper : DSE–2.3

**( Number Theory )**

*Full Marks* : 80

*Pass Marks* : 32

*Time* : 3 hours

1. *(a)* Define a linear Diophantine equation. 1

*(b)* If $p$ be a prime number and $a$ be any integer, then prove that either $p|a$ or $(a, p) = 1$. 2

*(c)* Find the general solution of $10x - 8y = 42$. 3

*(d)* If $a$, $b$, $c$ be integers such that $ac \equiv bc \pmod{m}$ and $d = (c, m)$, then show that

$$a \equiv b\left(\bmod \frac{m}{d}\right)$$ 4

*Or*

State and prove fundamental theorem of arithmetic.

2. (a) Define CRS $(\bmod m)$. 1

(b) What is the remainder when $5^{48}$ is divided by 12? 2

(c) Solve $6x \equiv 15 (\bmod 21)$. 3

(d) Prove that an integer $p$ is a prime if and only if

$$(p-1)! \equiv -1 (\bmod p)$$ 4

*Or*

State and prove Chinese remainder theorem.

3. (a) Define Möbius function. 1

(b) Find $\tau(n)$ and $\sigma(n)$, when $n = 12$. 3

(c) Prove that

$$\frac{\phi(n)}{n} = \sum_{d \mid n} \frac{\mu(d)}{d}$$ 3

*Or*

Find $\sum_{d \mid n} \phi(d)$ if $n = 12$.

(d) If $n$ be an integer >1, then show that $\tau(n)$ is odd $\Leftrightarrow n$ is a perfect square. 3

(e) If $f$ is a multiplicative arithmetic function, then prove that

$$g_1(n) = \sum_{d|n} f(d) \text{ and } g_2(n) = \sum_{d|n} \mu(d) f(d)$$

are both multiplicative arithmetic functions. 5

*Or*

If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ is the prime factorization of $n > 1$, then prove that

$$\sigma(n) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{k_2+1} - 1}{p_2 - 1} \cdot \dots \frac{p_r^{k_r+1} - 1}{p_r - 1}$$

and $\tau(n) = (k_1 + 1)(k_2 + 1) \dots (k_r + 1)$

4. (a) If $x$ and $y$ be any real numbers, then show that—

(i) $[x + m] = [x] + m$, $m$ be any integer;

(ii) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$. 2+2=4

( Turn Over )

*(b)* If $n$ be any integer $>2$, then prove that $\phi(n)$ is even. 3

*(c)* Evaluate $\phi(450)$. 3

*(d)* Let $a$ and $m > 0$ be integers such that $(a, m) = 1$, then prove that

$$a^{\phi(m)} \equiv 1 \pmod{m}$$ 5

*Or*

For each positive integer $n \geq 1$, then show that

$$n = \sum_{d \mid n} \phi(d)$$

the sum being extended over all positive divisors of $n$.

**5.** *(a)* If $a$ be an integer having exponent $h$ modulo $m$, i.e., $a^h \equiv 1 \pmod{m}$, then prove that

$$(a, m) = 1$$

that is $a$ and $m$ are coprime. 2

*(b)* If $a$ has exponent $h$ modulo $m$, then prove that $a^k$ has exponent $\frac{h}{d}$, where $d = (h, k)$. 4

(c) Find the primitive roots (mod 7). 4

(d) Define Legendre symbol and if $p$ be an odd prime and $a$, $b$ be any integers coprime to $p$, then prove that

$$\left(\frac{ab}{p}\right)=\left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$ 5

*Or*

Let $a$ be any odd integer, then show that

$$a^{2^{n-2}} \equiv 1 \pmod{2^n}, \quad \forall\, n \geq 3$$

6. (a) Evaluate the Legendre symbol $\left(-\frac{168}{11}\right)$. 2

(b) Show that the number of primes is not finite. 3

(c) Encrypt the message 'RETURN HOME' using Caesar cipher. 2

(d) If $x$, $y$, $z$ is a primitive Pythagorean triple, then show that one of the integers $x$ and $y$ is even, while the other is odd. 3

*(e)* If $p$ is an odd prime and $(a, p) = 1$, then show that $x^2 \equiv a \pmod{p^k}$, $k \geq 1$ has a solution if and only if $\left(\frac{a}{p}\right) = 1$. 5

*Or*

Show that an integer $a$ is a quadratic residue $\pmod{p}$ if and only if $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$.

Paper : DSE–2.4

**( Biomathematics )**

*Full Marks* : 80

*Pass Marks* : 32

*Time* : 3 hours

UNIT—I

1. Answer any *two* of the following questions : 7½×2=15

   *(a)* Explain Allee effect with suitable illustration.

   *(b)* Discuss the host-parasite problem by finding $dy/dx$ and sketching the trajectories in the $xy$-plane for the case when—

      *(i)* the birthrate equals the death rate in the host population;

      *(ii)* none of the parasites' eggs hatch.

   *(c)* The logistic differential equation is

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

( Turn Over )

Show that $N(t) = \frac{K}{1+Ce^{-rt}}$, where $C = \frac{K - N_0}{N_0}$, is a solution with initial condition $N(0) = N_0$. Here symbols have their usual meanings.

## UNIT—II

2. Answer any *two* of the following questions :

7½×2=15

(a) Show that in an SIR model with carriers who show no symptoms of the disease, the disease always remains endemic.

(b) Suppose that prey have a refuge from predators into which they can retreat. Assume that refuge can hold a fixed number of prey. How would you model this situation, and what predictions can you make?

(c) Explain the SIRS model with suitable illustrations.

UNIT—III

3. Answer any *two* of the following questions : 7½×2=15

(a) The interaction between two populations with densities $N_1$ and $N_2$ is modelled by

$$\frac{dN_1}{dt} = rN_1\left(1-\frac{N_1}{K}\right) - aN_1N_2(1-\exp[-bN_1])$$

$$\frac{dN_2}{dt} = -dN_2 + N_2e(1-\exp[-bN_1])$$

where $a$, $b$, $d$, $e$, $r$, $K$ are positive constants. What type of interaction exists between $N_1$ and $N_2$? What do the various terms imply ecologically?

(b) Write down all Routh-Hurwitz matrices $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{H}_3$ for the case of three species. Show that May's conditions are equivalent to the original Routh-Hurwitz criteria by evaluating the determinants of these matrices.

(c) Suppose $x$ is a predator and $y$ and $z$ both its prey. $z$ grows logistically in the absence of its predator. $x$ dies out in the absence of prey, and $y$ grows at an exponential rate in the absence of predator. Use Routh-Hurwitz techniques to examine whether these species can coexist in a stable equilibrium.

UNIT—IV

4. Answer any *two* of the following questions :

7½×2=15

(a) Discuss the stability of the following system :

$$x_{t+3} + 9x_{t+2} - 5x_{t+1} - 2x_t = 0$$

(b) Suppose a gene has 3 alleles in equilibrium in a randomly mating population. To find allele frequencies for the population, what is the minimum number of phenotype frequencies you must know? Answer the same question for $n$ alleles.

(c) Write a short note on blood flow in circulatory system.

UNIT—V

5. Answer any *two* of the following questions :

10×2=20

(a) Discuss the possibility of the existence of a stable age-structure, i.e., age-structure which does not change with time.

*(b)* Describe Nicholson-Bailey model for host parasite systems.

*(c)* Consider positive assortative mating, i.e., individuals mate only with those of like genotype. In contrast to random mating, set up a model for positive assortative mating. What conclusion do you reach?

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