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5 SEM TDC MTH M 2

2021

(March)

MATHEMATICS

(Major)

Course : 502

(Linear Algebra and Number Theory)

Full Marks : 80

Pass Marks : 32/24

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. (a) Let W be a vector subspace of a vector space V and $w \in W$, then what is $w + W$? 1

(b) In any vector space V, show that $(-\alpha)x = -(\alpha x)$ for each $\alpha \in \mathbb{R}$ and each $x \in V$. 2

2. (a) Answer any *two* from the following : 3×2=6

(i) Let $S = \{(a, b) : a, b \in \mathbb{R}\}$ and for $(a_1, b_1), (a_2, b_2) \in S$ and $c \in \mathbb{R}$, defined

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, 0)$$
$$c(a_1, b_1) = (ca_1, 0)$$

Show that S is not a vector space w.r.t. the operations defined above.

(ii) Prove that the intersection of subspaces of a vector space V is a subspace of V.

(iii) Show that $\{(1, 2), (4, 3)\}$ is a basis of $\mathbb{R}^2$.

(b) In the vector space $P_3(\mathbb{R})$ of all polynomials of degree ≤ 3 with real coefficients, prove that $2x^3 - 2x^2 + 12x - 6$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$. 5

Or

Let V be a vector space. Then prove that $\{v_1, v_2, \cdots, v_n\}$ is linearly dependent if and only if one of the v_i's is linear combination of the other v_j's.

(c) Show that $W = \{(x, y, z) \in \mathbb{R}^3 | y = z\}$ is a vector subspace of $\mathbb{R}^3$. Find a basis for W and hence find $\dim(W)$.

3+3=6

3. (a) What do you understand by affine subspace of a vector space? 2

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(x, y) = (x, y+3)$$

Determine whether T is linear. 3

(c) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$$ 3

4. Answer any *two* from the following : 6×2=12

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$$

Compute the nullity and rank of T.

(b) Let V and W be finite vector spaces of equal dimensions and let $T : V \to W$ be linear. Then prove that the following are equivalent :

(i) T is one-one

(ii) T is onto

(iii) Rank $T = \dim(V)$

(c) Let V and W be vector spaces over F and suppose that $\{v_1, v_2, \cdots, v_n\}$ is a basis for V. Prove that for $w_1, w_2, \cdots, w_n$ in W, there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, \cdots, n$.

GROUP—B

(Number Theory)

(*Marks* : 40)

5. (*a*) Write the law of trichotomy of natural numbers. 1

(*b*) Answer any *two* from the following : $3\times2=6$

(*i*) Prove that $ac|bc, c\neq 0$ gives $a|b$.

(*ii*) Prove that if x and y are odd, then x^2+y^2 is even but not divisible by 4.

(*iii*) Prove that two integers a and b are relatively prime if there exists integers x and y such that $1=ax+by$.

6. Answer any *two* from the following : $4\times2=8$

(*a*) Prove that any prime of the form $3n+1$ is also of the form $6m+1$.

(*b*) Prove that there are arbitrary large gaps in the series of primes. Hence list 5 consecutive composite numbers.

(*c*) Find the number of zeroes with which the decimal representation of 50! terminates.

7. (a) State Wilson's theorem. 1

(b) If $a \equiv b \pmod{m}$, $c \equiv d \pmod{m}$, then prove that $ac \equiv bd \pmod{m}$. 3

(c) Find the general solution in integers $3x + 5y = 1$. 3

8. Answer any *two* from the following : 4×2=8

(a) Solve the following :

$$x \equiv 1 \pmod{3}$$
$$x \equiv 1 \pmod{5}$$
$$x \equiv 1 \pmod{7}$$

(b) Prove that if $\gcd(a, 35) = 1$, then $a^{12} \equiv 1 \pmod{35}$.

(c) Using Fermat's theorem, find the unit digit of 3^{400}.

9. (a) Write the value of $\sigma(5)$. 1

(b) Answer any *three* from the following : 3×3=9

(i) Prove that

$$\prod_{d \mid n} d = n^{\frac{1}{2} d(n)}$$

where $d(n)$ denotes the number of positive divisors of n.

(ii) If p is prime, show that

$$\phi(p) + \sigma(p) = p\,d(p)$$

(iii) Find the value of $\sigma_2(8)$.

(iv) Define Euler's function $\phi(n)$ for $n \in \mathbb{N}$. For any prime p, find the value of $\phi(p)$.

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