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Recent Trends of Mathematics & its Applications

Editor

Sahin Ahmed

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EBH Publishers (India)
Guwahati-1

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First Published in 2014 by
EBH Publishers (India)
an imprint of Eastern Book House
136, M.L. Nehru Road, Panbazar
Guwahati-781 001 (India)

Phone : +91 361 2513876, 2519231, 92070 45352
Fax : +91 361 2519231
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Foreward

With higher education becoming an international service, there is a growing concern all over the world about quality, standard and recognition. This has made the concerned authority to evolve quality bench markings for ascertaining and assuring quality at different levels of higher education.

In line with these elements, the UGC is providing scheme for financial Assistance to institutions for organizing conferences, workshop and seminars at state, national and international levels in various fields. Rajiv Gandhi University is one of the regular recipients of this financial assistance and has organized workshops/seminars/conferences at national and international levels for a number of times.

This proceeding of national conference on "*Recent Trends of Mathematics and its Applications*" is a collection of research papers written by different scholars for RTMA-14 organized by the Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh, Arunachal Pradesh. I do believe that the proceeding would be useful to researchers who pursue mathematical sciences with different applications and would serve as a theoretical guide to experimental survey for getting an important and reliable data.

I offer my grateful acknowledgements to all the faculty members of the Department of Mathematics, Rajiv Gandhi University for having the idea of publishing the proceedings of the conference in a book form.

21/5/014

(Prof. Tamo Mibang)
Vice Chancellor,
Rajiv Gandhi University

Preface

In the recent days, many mathematicians are working toward the extension of existing mathematical methods, thus contributing to the development of both pure and applied mathematics. In the last decades we have witnessed an explosion of knowledge in all sciences, leading to a tremendously increasing need of mathematical tools. That explains the great development of different areas in applied mathematics, such as applied differential equations, applied functional analysis, applied statistics, financial mathematics, mathematical biology, numerical analysis and variational methods, and even the creation of new areas, such as Computational fluid dynamics, cryptology, mathematical psychology and mathematical sociology and mathematics education.

Many theoretical advances in mathematics have been reported in the last 20 years, in both classical and modern fields of mathematics. It is enough to look at the work of the recipients of different awards, such as the *Wolfe Prize*, the *Fields Medal* and the *Abel Prize* —equivalent to the *Nobel Prize*— established in 2002 by the *Niels Henrik Abel Memorial Fund*. The *Abel Prize* laureates are honored for their noteworthy work across a broad spectrum of areas. Examples of these awards include: (2003), Jean-Pierre Serrre (France), for playing a key role in shaping the modern form of many parts of mathematics, including topology, algebraic geometry and number theory; (2005), Peter D. Lax (Hungary/USA), for his groundbreaking contributions to the theory and application of partial differential equations and to the computation of their solutions. Other some land marking contributions are;

- 2004: Michael F. Atiyah (UK/Lebanon) and Isadore M. Singer (USA), for their discovery and proof of the index theorem, bringing together topology, geometry and analysis, and their outstanding role in building new bridges between mathematics and theoretical physics;
- 2006: Lennart Carleson (Sweden), for his profound and seminal contributions to harmonic analysis and the theory of smooth dynamical systems;

- 2007: S.R. Srinivasa Varadhan (India/USA), for his fundamental contributions to probability theory and, in particular, for creating a unified theory of large deviation;
- 2008: John G. Thompson (USA) and Jacques Tits (Belgium/France), for their profound achievements in algebra and, in particular, for shaping modern group theory;
- 2009: Mikhail Gromov (Russia/France), for his revolutionary contributions to geometry;
- 2010: John Tate (USA), for his vast and lasting impact on the theory of numbers.

The book "*Recent Trends of Mathematics and its Applications*" is the result of 15 selected Research papers of the Conference proceedings which are being edited to fulfill quality oriented objective of the Conference. Editing a book like this is always complicated but challenging too, as the contributors belong to different parts of North-East and other parts of India. I have to go through many stages in shaping this work into a book format which needs lot of time and patience. In the process I had to follow certain criterion while selecting the papers because of which it was not possible for us to select all the papers for publication. I sincerely regret that, while selecting the papers we had to approach many of the experts of our Editorial Board in the field for their comments, suggestions and recommendations. I tried our best to fulfil the objective of the venture in all possible way.

This Proceedings contains the substance of contributed research Presentations delivered at the National Conference on *Recent Trends of Mathematics and its Applications* (RTMA-2014), which was held in the Rajiv Gandhi University, Rono Hills, Doimukh, Arunachal Pradesh, India during 26th and 27th May 2014.

The main object of the seminar is to bring together eminent researchers of various fields of Mathematics like Algebra, Aanalysis, Fluid Mechanics, Number theory, Wave-lets analysis, Cryptography, Graph theory, Fuzzy sets and Fuzzy logics, Relativity, Numerical analysis, Differential equations, Integral Transforms, Mathematics education, Probability and Statistics and its recent trends of applications.

At the outset I take this opportunity to express my gratefulness to *Dr. Bipon Hazarika* (Convenor, RTMA) and *Dr. Saifur Rahman* (Secretary, RTMA), Department of Mathematics, Rajiv Gandhi University for providing invaluable and thought provoking suggestions and constant encouragement in every step of completion of my work.

I gratefully acknowledge the inestimable inspiration and suggestions I received from the Editorial members *Prof. H. Saikia*, *Prof. N. Ahmed* and *Prof. H. K. Sarmah* of Gauhati University, Guwahati, and *Dr. Utpal Bhattachargee*, Dept. of CSE, R. G. U.

I acknowledge my gratefulness to the contributors and experts for their kind cooperation's with us during the time of editing of the book. I also take this opportunity to offer our thanks to honourable Vice Chancellor *Prof. Tamo Mibang*, The Registrar *Dr. Rachob Taba*, and The COE *Prof. R. Tamuli* of Rajiv Gandhi University for their sincere cooperation and suggestions. I also record our gratitude to *Mr. J. P. Sharma*, EBH publishers (India), Guwahati for his kind consent to publish the book in time. I sincerely call upon one and all to bear with us for the inadvertent omission and commission if any in the book.

Finally, I take this opportunity to acknowledge the help, guidance and cooperation received by me from several individuals and institutions/ organizations while carrying out the present work.

Editor

Acknowledgement

It gives us immense pleasure to publish a book entitles "*Recent Trends of Mathematics and its Applications*". Our heartfelt thanks to the Rajiv Gandhi University, Rono Hills, Arunachal Pradesh for providing fund for organizing National Conference which made the publication of this book possible. We express our gratitude to the contributors of papers, without their hard work and tireless effort, the publication of this book is impossible.

We are very thankful to EBH Publishers (India) Guwahati, for their full cooperation in publishing this book.

We are the sole responsible persons for all the errors, mistakes and failure of this book. Any suggestion and advice for the betterment and upgradation is heartily welcome.

May 26, 2014

Dr. Bipan Hazarika
Dr. Saifur Rahman
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1

Thermal Radiation Effect on Convective Heat Transport in a Porous Media under Laplace Transform

Karabi Kalita

Abstract

Magnetohydrodynamic boundary layer flow and heat transfer through a *Darcian* porous medium bounded by a uniformly moving semi-infinite isothermal vertical plate in presence of thermal radiation is presented. In this analysis, we considered the flow is of an unsteady, viscous, incompressible, electrically-conducting Newtonian fluid which is an optically thin gray gas. Suitable transformations are used to convert the partial differential equations corresponding to the momentum and energy equations into nonlinear ordinary differential equations. Analytical solutions of these equations are obtained by *Laplace transform*. The effects of Hartmann number (M), porosity parameter (K), thermal radiation parameter (R_a), and Prandtl number (Pr) on flow velocity, fluid temperature, velocity and temperature gradients at the surface are studied graphically. Velocity is reduced with Hartmann number but enhanced with thermal radiation and porosity parameter. An increase in porosity/thermal radiation parameter is found to strongly enhance flow velocity values. Velocity gradient at y=0 is increased with porosity parameter. Applications of the study arise in engineering and geophysical sciences like magnetohydrodynamic transport phenomena and magnetic field control of materials processing, solar energy collector systems.

Keywords: *Optically thin gray gas; Hartmann number; Porous media; Heat Transport; Unsteady boundary layer Flow.*

Introduction

Fluid flow through a porous media has been studied theoretically and experimentally by numerous authors due to its wide applications in various fields such as diffusion technology, transpiration cooling, hemodialysis processes, flow control in nuclear reactors, etc. In view of geophysical applications of the flow through porous medium, a series of investigations has been made by Raptis *et.al* [1-2], where the porous medium is either bounded by horizontal, vertical surfaces or parallel porous plates. Singh *et.al* [3] and Lai and Kulacki [4] have been studied the free convective flow past vertical wall. Nield [5] studied convection flow through porous medium with inclined temperature gradient. Singh *et al.* [6] also studied periodic solution on oscillatory flow through channel in rotating porous medium. Further due to increasing scientific and technical applications on the effect of radiation on flow characteristic has more importance in many engineering processes occurs at very high temperature and acknowledge radiative heat transfer such as nuclear power plant, gas turbine and various propulsion devices for aircraft, missile and space vehicles. The effect of radiation on flow past different geometry a series of investigation have been made by Hassan [7], Seddeek [8] and Sharma *et al* [9]. The combined radiation–convection flows have been extended by by Ghosh and Beg [11] to unsteady convection in porous media. Hossain and Takhar [21] studied the mixed convective flat plate boundary-layer problem using the Rosseland (diffusion) flux model. Mohammadein *et al*. [22] studied the radiative flux effects on free convection in the Darcian porous media using the Rosseland model. The transient magnetohydrodynamic free convective flow of a viscous, incompressible, electrically conducting, gray, absorbing–emitting, but non-scattering, optically thick fluid medium which occupies a semi-infinite porous region adjacent to an infinite hot vertical plate moving with a constant velocity is presented by Ahmed and Kalita [13]. Raptis and Perdikis [14] have also studied analytically the transient convection in a highly porous medium with unidirectional radiative flux. Ghosh and Pop [15] studied indirect radiation effects on convective gas flow. Ahmed and Kalita [16] investigated the effects of chemical reaction as well as magnetic field on the heat and mass transfer of Newtonian two-dimensional flow over an infinite vertical oscillating plate with variable mass diffusion. Ahmed [17] presented the effects of conduction-radiation, porosity and chemical reaction on unsteady hydromagnetic free convection flow past an impulsively-started semi-infinite vertical plate embedded in a porous medium in presence of thermal radiation. The thermal radiation and Darcian drag force MHD unsteady thermal-convection flow past a semi-infinite vertical plate immersed in a semi-infinite saturated porous regime with variable surface temperature in the presence of transversal uniform magnetic field have been discussed by Ahmed *el al*. [18].

The present paper is to investigate the effect of magnetic field and radiation on unsteady free convection heat transfer flow of viscous laminar electrically conducting Newtonian radiating fluid past an impulsively startedsemi-infinite vertical surface in a Darcian porous medium. The analytical solution is obtained using Laplace Transform technique and discussed graphically for various flow parameters.

Mathematical formulation

Considering the magnetohydrodynamic unsteady free convection and heat transfer flow of a viscous, incompressible, electrically conducting Newtonian fluid past a semi-infinite isothermal vertical plate embedded in a porous media under the influence of the thermal buoyancy. A uniform magnetic filed of uniform strength B_0 is assumed to be applied normal to the surface. The flow is assumed to be in the $\bar{x}$-direction, which is taken

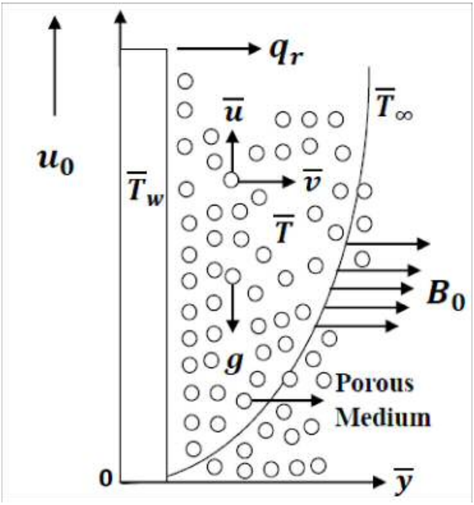

Fig.1.1.Physical configuration and co-ordinate system

along the plate in the upward direction and $\bar{y}$ -axis is normal to it. Initially it is assumed that the plate and the fluid are at the same temperature $\bar{T}$. At time $t > 0$, the plate temperature is instantly raised to $\bar{T}_w > \bar{T}_\infty$ and, which is thereafter maintained constant, where $\bar{T}_\infty$ is the temperature outside the boundary layer. The ***induced magnetic field*** and ***viscous dissipation*** is assumed to be negligible as the ***magnetic Reynolds number*** of the flow is taken to be very small. Assuming that the Boussinesq and boundary-layer approximations hold, the governing equations to the problem are given by:

$$\frac{\partial \bar{u}}{\partial \bar{t}} = g\beta(\bar{T} - \bar{T}_\infty) + \nu \frac{\partial^2 \bar{u}}{\partial^2 \bar{y}} - \frac{\sigma B_0^2}{\rho}\bar{u} - \frac{\nu}{\bar{K}}\bar{u} \tag{1}$$

$$\rho C_P \frac{\partial \bar{T}}{\partial \bar{t}} = \kappa \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} - \frac{\partial q_r}{\partial \bar{y}} \tag{2}$$

The initial and boundary conditions are

$$\begin{aligned} &\bar{t} \le 0 : \bar{u} = 0, \bar{T} = \bar{T}_\infty \text{ for all } \bar{y} \le 0 \\ &\bar{t} > 0 : \bar{u} = u_0 \ , \bar{T} = \bar{T}_w \text{ at } \bar{y} = 0 \\ &\bar{t} > 0 : \bar{u} \to 0, \bar{T} \to \bar{T}_\infty, \text{ as } \bar{y} \to \infty \end{aligned} \tag{3}$$

The local radiant absorption for the case of an optically thin gray gas is expressed (Cogley *et al*. [19]) as

$$\frac{\partial q_r}{\partial \bar{y}} = -4\bar{a}\,\bar{\sigma}(\bar{T}_\infty^4 - \bar{T}^4) \tag{4}$$

Where $\bar{\sigma}$ and $\bar{a}$ are the ***Stefan-Boltzmann constant*** and ***mean absorption co-efficient*** respectively. We assume that the differences within the flow are sufficiently small so that $\bar{T}^4$ can be expressed as a linear function of $\bar{T}$ after using Taylor's series to expand about the free stream temperature and neglecting higher order terms. This results in the following approximation:

$$\overline{T}^4 \cong 4\overline{T}_\infty^3 \overline{T} - 3\overline{T}_\infty^4 \tag{5}$$

$$\rho C_P \frac{\partial \overline{T}}{\partial \overline{t}} = \kappa \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} - 16\overline{a}\,\overline{\sigma}\,\overline{T}_\infty^3 (\overline{T}_\infty - \overline{T}) \tag{6}$$

Introducing the following non-dimensional quantities:

$$u = \frac{\overline{u}}{u_0},\ t = \frac{\overline{t} u_0^2 G}{\nu},\ y = \frac{\overline{y} u_0 \sqrt{G}}{\nu},\ \theta = \frac{\overline{T} - \overline{T}_\infty}{\overline{T}_w - \overline{T}_\infty},\ K = \frac{u_0^2\, G \overline{K}}{\nu^2}, \tag{7}$$

$$G = \frac{g\beta\nu\left(\overline{T}_w - \overline{T}_\infty\right)}{u_0^3},\ R_a = \frac{16\overline{a}\,\overline{\sigma}\nu^2\sigma\overline{T}_\infty^3}{\kappa u_0^2},\ M = \frac{\sigma B_0^2 \nu}{\rho u_0^2 G},\ \Pr = \frac{\mu C_p}{\kappa}$$

Using the transformations (7), the non-dimensional forms (1), (3) and (6) are

$$\frac{\partial u}{\partial t} = \theta + \frac{\partial^2 u}{\partial y^2} - (M + K^{-1})u \tag{8}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2} - \frac{R_a}{Pr}\theta \tag{9}$$

The corresponding initial and boundary conditions transformed to:

$$\begin{aligned} &u = 0,\ \theta = 0 \ \text{ for all } t,\ y \le 0 \\ &t > 0:\ u = 1, \theta = 1 \ \text{ at } y = 0 \\ &t > 0: u \to 0,\ \theta \to 0 \ \text{ as } y \to \infty \end{aligned} \tag{10}$$

Method of Solution

The unsteady, non-linear, coupled partial differential equations (8) and (9) along with their boundary conditions (10) have been solved analytically using *Laplace transforms technique* and the solutions are as follow

$$u(y,t)=\frac{1}{2}\left[\begin{array}{l}\left(1-\frac{1}{\psi}\right)\left\{\begin{array}{l}e^{2\eta\sqrt{\xi t}}erfc\left(\eta+\sqrt{\xi t}\right)\\+e^{-2\eta\sqrt{\xi t}}erfc\left(\eta-\sqrt{\xi t}\right)\end{array}\right\}\\+\frac{1}{\psi}e^{\lambda t}\left\{\begin{array}{l}e^{2\eta\sqrt{(\xi+\lambda)t}}erfc\left(\eta+\sqrt{(\xi+\lambda)t}\right)\\+e^{-2\eta\sqrt{(\xi+\lambda)t}}erfc\left(\eta-\sqrt{(\xi+\lambda)t}\right)\end{array}\right\}\\+\frac{1}{\psi}\left\{\begin{array}{l}e^{2\eta\sqrt{\Pr R_a t}}erfc\left(\eta\sqrt{\Pr}+\sqrt{R_a t}\right)\\+e^{-2\eta\sqrt{\Pr R_a t}}erfc\left(\eta\sqrt{\Pr}-\sqrt{R_a t}\right)\end{array}\right\}\\-\frac{1}{\psi}e^{\lambda t}\left\{\begin{array}{l}e^{2\eta\sqrt{\Pr(R_a+\lambda)t}}erfc\left(\eta\sqrt{\Pr}+\sqrt{(R_a+\lambda)t}\right)\\+e^{-2\eta\sqrt{\Pr(R_a+\lambda)t}}erfc\left(\eta\sqrt{\Pr}-\sqrt{(R_a+\lambda)t}\right)\end{array}\right\}\end{array}\right] \quad (11)$$

$$\theta(y,t)=\frac{1}{2}\left\{\begin{array}{l}e^{2\eta\sqrt{\Pr R_a t}}erfc\left(\eta\sqrt{\Pr}+\sqrt{R_a t}\right)\\+e^{-2\eta\sqrt{\Pr R_a t}}erfc\left(\eta\sqrt{\Pr}-\sqrt{R_a t}\right)\end{array}\right\} \quad (12)$$

Skin friction and Nusselt Number

The non-dimensional skin friction and Nusselt number is given as follows:

$$\tau=-\left[\frac{\partial u(y,t)}{\partial y}\right]_{y=0}$$

$$\begin{aligned}&=\left(1-\frac{1}{\psi}\right)\left\{\frac{e^{-\xi t}}{\sqrt{\pi t}}+\sqrt{\xi}erf\left(\sqrt{\xi t}\right)\right\}\\&+\frac{1}{\psi}e^{\lambda t}\left\{\frac{e^{-(\xi+\lambda)t}}{\sqrt{\pi t}}+\sqrt{(\xi+\lambda)}erf\left(\sqrt{(\xi+\lambda)t}\right)\right\}\\&+\frac{1}{\psi}\sqrt{\Pr}\left\{\frac{e^{-R_a t}}{\sqrt{\pi t}}+\sqrt{R_a}erf\left(\sqrt{R_a t}\right)\right\}\end{aligned} \quad (13)$$

$$-\frac{1}{\psi}\sqrt{\Pr}e^{\lambda t}\left\{\frac{e^{-(R_a+\lambda)t}}{\sqrt{\pi t}}+\sqrt{(R_a+\lambda)}erf\left(\sqrt{(R_a+\lambda)t}\right)\right\}$$

$$Nu=-\left[\frac{\partial\theta(y,t)}{\partial t}\right]_{y=0}$$

$$=\sqrt{\Pr}\left\{\frac{e^{-R_a t}}{\sqrt{\pi t}}+\sqrt{R_a}erf\left(\sqrt{R_a t}\right)\right\} \tag{14}$$

Results and Discussion

The problem of thermal radiation effect on a porous media transport under optically thick approximation formulated, analyzed and solved analytically. In order to point out the effects of physical parameters namely; magnetohydrodynamic force (M), radiation parameter (R_a), Porosity parameter (K) on the flow patterns, the computation of the flow fields are carried out. The values of velocity, temperature, shear stress and rate of heat transfer are obtained for the physical parameters as mention. The velocity profiles has been studied and presented in Figs. 1.2 to 1.4. Figure 1.2 shows the effect of the Hartmann number M on the fluid velocity and the results show that the presence of the magnetic force causes retardation of the fluid motion represented by general decreases in the fluid velocity. It is due the fact that magnetic force which is applied in the normal direction to the flow produces a drag force which is known as *Lorentz force*. The opposite trend is observed in Fig. 1.3 for the case when the value of the porous permeability (K = 0.2, 0.5, 1.0, 1.5) is increased. As depicted in this figure, the effect of increasing the value of porous permeability is to increase the value of the velocity component in the boundary layer due to the fact that drag is reduced by increasing the value of the porous permeability on the fluid flow which results in increased velocity. The trend shows that the velocity is accelerated with increasing porosity parameter. The effect of velocity for different values of radiation (R_a =0, 15, 16, 18) is also presented in Fig. 1.4. It is then observed that the flow velocity is accelerated with the small value R_a = 0.0, the values of flow velocity reduces exponentially from the plate, while for the higher values of R_a the flow velocity has a bigger pick in the neighbourhood of $y = 0.2$, but the opposite behaviour has been observed for the effects of magnetohydrodynamic force.

The temperature profiles are calculated for different values of thermal radiation parameter (R_a=0, 5, 10, 15) at time $t = 0.2$ and these are shown in Fig. 1.5. The effect of thermal radiation parameter is important in temperature profiles. It is observed that the temperature increases with decreasing radiation parameter. Figure 1.6 reveals temperature variations with Pr (Prandtl number) which signifies the ratio of momentum to thermal diffusivity at $t = 0.2$. The temperature is observed to decrease with an increase in Pr. For lower Pr fluids, heat diffuses faster than momentum and vice versa for higher Pr fluids. Larger Pr values correspond to a thinner thermal boundary layer thickness and more uniform temperature distributions across the boundary layer. Smaller Pr fluids possess higher thermal conductivities so that heat can diffuse away from the vertical surface faster than for higher Pr fluids (low Pr fluids correspond to thicker boundary layers). For working oils ($Pr = 11.4$), convection is very effective in transferring energy from an area, compared to pure conduction and momentum diffusivity is dominant. It is also observed that the temperature is maximum near the plate and decreases away from the plate and finally takes asymptotic value for all values of Pr.

Figure 1.7 illustrates the transient shear stress variation with Hartmann number and radiation parameter. The shear stresses at the wall are seemed to be enhanced with a rise in Hartmann number, which is proportional to the square of the magnetic field, B_0. A reversed trend has been observed for conduction-radiation on shear stress (τ) i.e. τ decreases substantially at the wall for $R_a = 0, 8, 10, 11$. For the non-radiating flow case, $R_a = 0$, a significant linear flow of shear stress is sustained against hydromagnetic force. For the case, $R_a = 10, 11$, a significant flow reversal (backflow) is obtained within the region $0<M<2.5$ i.e. shear stresses become negative. However for $R_a = 0, 8$, all backflow is eliminated entirely from the regime for all hydromagnetic forces and only positive shear stresses arise at the plate.

Figure 1.8 shows the distribution of shear stress at the wall for various porosity parameters over time. With a rise in radiation parameter, K, from 0.5, 1.0 through 1.5 to 2.0 decreases the magnitude of the shear stress through the boundary layer. We also observe that for all values of K, shear stress remains positive i.e. no flux reversal arises for all times into the boundary layer. With progression in time, t, the shear stress is however found to decrease continuously. Finally, in Fig. 1.9 the distribution of rate of heat transfer with radiation parameter is shown against t. Inspection shows that, increasing radiation parameter, R_a, tends to boost the heat transfer rate at the wall i.e. elevate Nu magnitudes. A substantial decrease is observed in Nu for the time parameter.

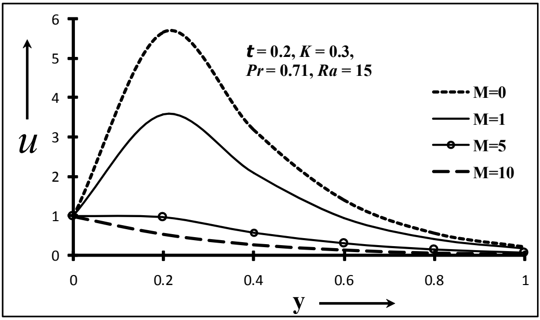

Fig.1.2 Flow velocity profile for *M*

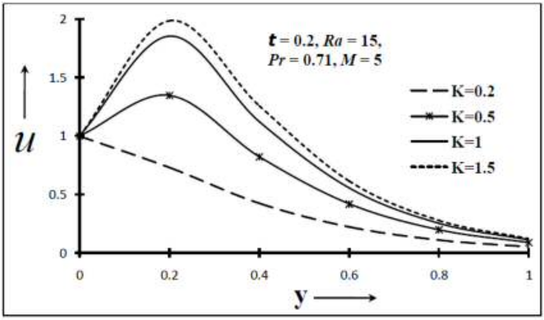

Fig.1.3 Flow velocity profile for *K*

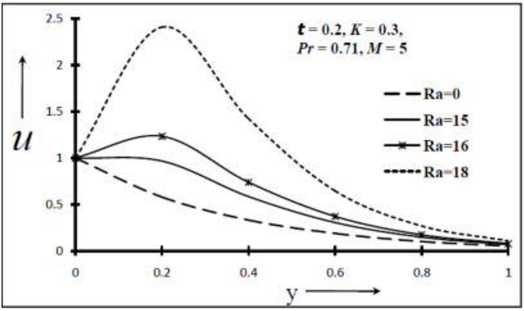

Fig. 1.4 Flow velocity profile for R_a

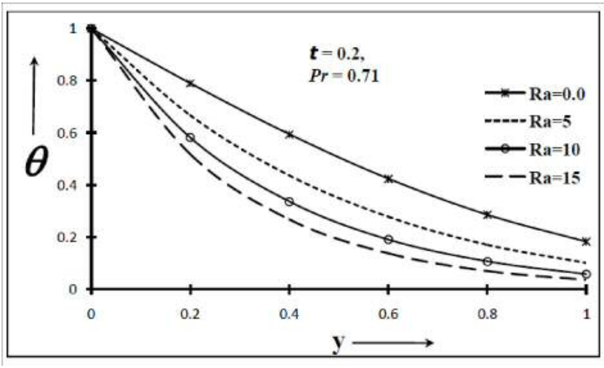

Fig. 1.5 Temperature profile for R_a

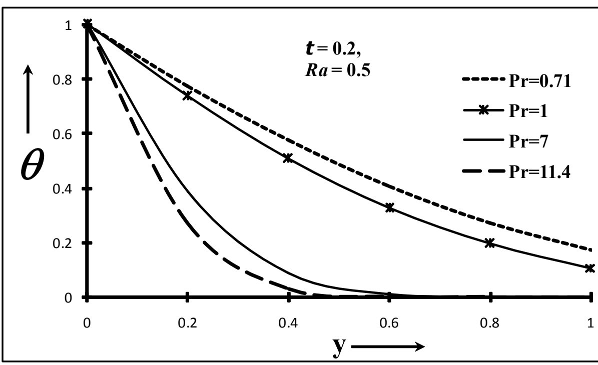

Fig. 1.6 Temperature profile for *Pr*

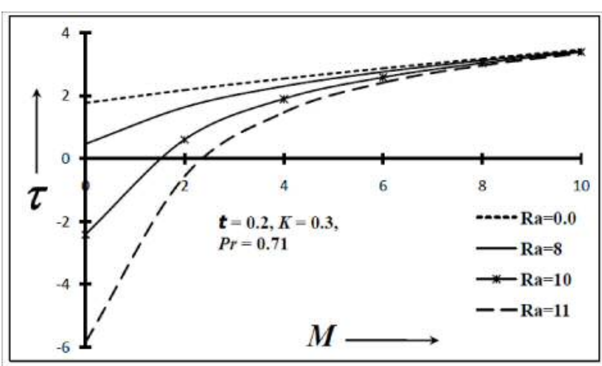

Fig. 1.7 Shear stress variation for R_a

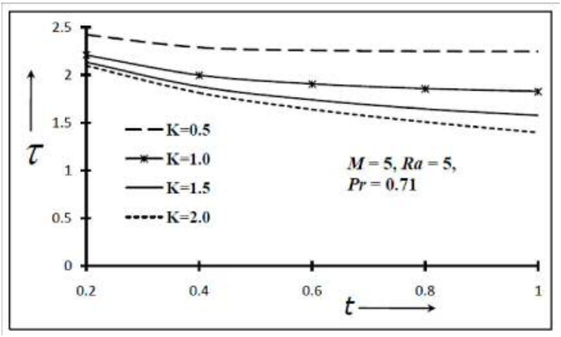

Fig. 1.8 Shear stress variation for *K*

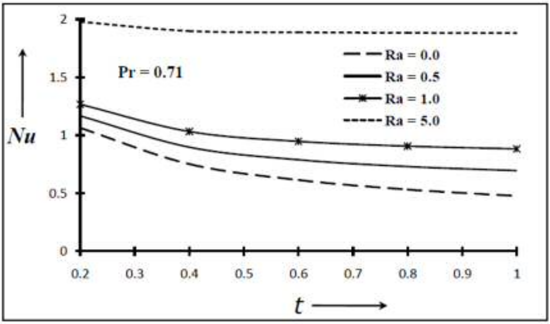

Fig. 1.9 Nusselt no. variation for R_a

Conclusion

In the present work, we have analyzed flow, heat transfer on convection flow of a viscous incompressible, electrically conducting and radiating fluid over an infinite vertical plate embedded in a Darcian porous regime in the presence of transverse magnetic field and thermal radiation using the classical model for the radiative heat flux. Final results are computed for variety of physical parameters which are presented by means of graphs. *Laplace transforms* solutions for the non-dimensional momentum and energy equations subject to transformed boundary conditions have been obtained. The flow has been shown to be decelerated with increasing Hartmann number but accelerated with conduction-radiation and porosity parameters. Increasing Hartmann number also increases the shear stress and back flow has been observed for higher radiation near the wall. A positive decrease in R_a or K strongly enhanced the shear stress. The study has important applications in materials processing and nuclear heat transfer control, as well as MHD energy generators.

References

1. Raptis, A., Tzivanidis,G. and Kafousias,N. Free convection and mass transfer flow through a porous medium bounded by an infinite vertical limiting surface with constant suction. Letters Heat Mass Transfer, 8, 417-424, (1981).
2. Raptis, A., Kafousias, N. and Massalas, C. Free convection and mass transfer flow through a porous medium bounded by an infinite vertical porous plate with constant heat flux, *ZAMM*, **62**, 489-491 (1982).
3. Singh, P., Misra,J. K. and Narayan,K. A. Free convection along a vertical wall in a porous medium with periodic permeability variation, *Int. J. Numer. Anal. Methods Geometh*. **13**, 443-450 (1989).
4. Lai, F. C. and Kulacki, F. A. The effect of variable viscosity on convective heat transfer along a vertical surface in a saturated porous media, *Int. J. Heat Mass Transfer*. **33**, 1028-1031 (1990).
5. Nield, D. A. Convection in a porous medium with inclined temperature gradient: An additional results. Int. J. Heat Mass
6. Singh, K. D., Gorla, M. G. and Hans Raj, A periodic solution of oscillatory Couette flow through porous medium in rotating system, *Indian, J. Pure Appl. Math*., **36**(3), 151-159 (2005).
7. Hassan, A. M. and El-Arabawy. Effect of suction/injection on the flow of a micropolar fluid past a continuously moving plate in the presence of radiation, *Int. J. Heat Mass Transfer*, **46**(8), 1471-1477, (2003).
8. Seddeek, M. A. The effect of variable viscosity on hydromagnetic flow and heat transfer past a continuously moving porous boundary with radiation, *Int. Comm., Heat Mass Transfer*, **27**(7), 1037-1046 (2000).
9. Sharma, B. K. Sharma, P. K. and Tara Chand. Effect of radiation on temperature distribution in three-dimensional Couette flow with heat source/sink, *Int. J. of Applied Mechanics and Engineering*, 16(2), 531-542 (2011).
10. Ghosh, S. K. and Be2 g, O. A. Theoretical analysis of radiative effects on transient free convection heat transfer past a hot vertical surface in porous media, *Nonlinear Anal.: Model. Control*, **13**(4), 419–432 (2008).
11. Hossain, M. A. and Takhar, H. S. Radiation effects on mixed convection along a vertical plate with uniform surface temperature, *Heat Mass Transfer*, **31**, 243–248 (1996).
12. Mohammadein, A. A., Mansour, M. A., El Gaied, S. M. and Gorla, R. S. R. Radiative effect on natural convection flows in porous media, *Transport Porous Media*, **32**(3), 263–283 (1998).

13. Ahmed, S. and Kalita, K. Magnetohydrodynamic transient flow through a porous medium bounded by a hot vertical plate in the presence of radiation: a theoretical analysis, *Journal of Engineering Physics and Thermophysics,* **86**(1), 30-39 (2013).

14. Raptis, A. and Perdikis, C. Unsteady flow through a highly porous medium in the presence of radiation, *Transport Porous Media J*., **57**(2), 171–179 (2004).

15. Ghosh, S. K. and Pop, I. Thermal radiation of an optically thin gray gas in the presence of indirect natural convection, *Int. J. Fluid Mechanics Research*, 34(6), 515–520 (2007).

16. Ahmed, S. and Kalita, K. Analytical numerical study for MHD radiating flow over an infinite vertical plate bounded by porous medium in presence of chemical reaction, *Journal of Applied Fluid Mechanics*, **6**(4), 597-607 (2013).

17. Ahmed, S. Numerical analysis for magnetohydrodynamic chemically reacting and radiating fluid past a non-isothermal uniformly moving vertical surface adjacent to a porous regime, *Ain Shams Engineering Journal*, Elsevier, In Press (2014).

18. Ahmed, S., Batin, A. and Chamkha, A. J. Finite Difference Approach in porous media transport modeling for Magnetohydrodynamic unsteady flow over a vertical plate: Darcian Model, *Int. J. of Numerical Methods for Heat and Fluid Flow*, **24**(5), 1204-1223 (2014).

19. Cogley A. C., Vinceti W. G. and Gilles S. E. Differential approximation for radiation transfer in a nongray near equilibrium, *AIAA Journal*, **6**, 551-553 (1968).

2

Orbit of a test charged particle inside Kerr-Newman de/anti de Sitter Black Hole

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K.Yugindro Singh

Abstract

In this paper we obtain the geodesic orbit equation of a test charged particle in the Kerr-Newman anti de Sitter black hole by using the Hamilton-Jacobi equation. It appears that there are stable circular orbits of a charged particle within the inner horizon and that the combined effect of the charge and rotation of the Kerr-Newman de/anti de Sitter black hole and the coupling between the charge of the test particle and the electromagnetic field of the black hole may account for this..

Keywords: *Kerr-Newman black hole; event horizons; circular orbits.*

Introduction

In Einstein's general relativity, it is not possible to describe circular geodesics of arbitrary radii; there is a minimum radius below which no circular orbits are possible. The general types of particle orbits in the black hole gravitational field have been considered by Chandrasekhar [1]: orbits of the first kind, which are completely confined outside the black hole event horizon, and orbit of the second kind, which penetrates inside the black hole. It has been shown in the case of rotating or charged black hole that there are bound planetary orbits, known as the orbits of third kind, which neither come out nor terminate at the central singularity [2]. The third kind of bound orbits inside the inner horizon were discussed in [3, 4] for a test charged particle around a rotating charged black hole and in [5] for the neutral particle.

The geometrical structure of a Kerr-Newman black hole is asymptotically de sitter at large times when a repulsive cosmological constant, $\Lambda > 0$ is considered and may contain a cosmological horizon with a dynamic background and for an attractive cosmological constant, $\Lambda < 0$ the geometrical structure is asymptotically anti de Sitter and contains black hole horizons. The recent serious study of the observational and theoretical aspects of a small positive cosmological constant term which may be relevant at the present epoch [6] and the accurate measurements of the anisotropy of the cosmic relic background radiation and the observational analysis of type I_a supernova with high red shift parameter Zd" 1 in the framework of the inflationary cosmology [7], suggest that a repulsive cosmological constant Ë > 0 has to be taken seriously for understanding the properties of the presently observed universe. On the other hand, it is recognized that the de/anti de Sitter space-time has an important role in the multidimensional string theory [8].

The geometrical properties of the Kerr-Newman de/anti de sitter space-time with non zero cosmological constant are described by the geodesic equations of motion of a test particle. The motion of a test charged particle in the gravitational field of a charged black hole is fully described by three integrals of motion namely, E, the total particle energy, $L_{ö}$, the azimuthal component of the angular momentum and Q, the Carter constant[9].

In section 2, we review general geodesic orbits of test particle in the Kerr Newman de/anti de sitter space-time. In section 3, we discuss the singularities, event horizons and geometrical surfaces of Kerr Newman de/anti de sitter space-time. In section 4, the possibility for bound stable periodic orbits for a charged particle inside the inner horizon is discussed. Section 5 is a brief conclusion. We use the units G = c = 1 throughout the paper.

Methods

General geodesic orbits in the Kerr-Newman de/anti de Sitter space-time

The equations of motion of a test particle of mass m and charge ε in the gravitational field of a rotating charged black hole in the Kerr-Newman de/anti Sitter space-time may be determined from the principle of least action with the action defined in [10]:

$$S = \int_a^b (-m^2 + \varepsilon A_\mu \dot{x}^\mu) d\lambda \tag{1}$$

where $d\lambda = \frac{1}{m} d\tau$, $d\tau = \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$ being the proper time of the test charged particle along geodesics; $A_\mu = (-\phi, \mathbf{A})$ is the covariant four vector potential and a dot overhead a symbol denotes differentiation with respect to the parameter l. From the action, the Lagrangian is identified as

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \varepsilon A_\mu \dot{x}^\mu \tag{2}$$

with the normalizing condition,

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -m^2 \tag{3}$$

Effecting the variation of the action, one obtains

$$\delta S = (g_{\mu\nu} \dot{x}^\nu + \varepsilon A_\mu) \delta x^\mu \Big|_a^b + \int_a^b [-m \frac{du_\mu}{d\tau} + \varepsilon F_{\mu\nu} u^\nu] \delta x^\mu d\tau \tag{4}$$

where $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$ is the electromagnetic field tensor and $u^\nu = \frac{dx^\nu}{d\tau}$. The integrated term vanishes at both limits as the end points are fixed. According to the principle of least action, $\delta S = 0$ for the correct path of motion of the particle. This condition leads to the following equations of motion for the test charged particle in the given field:

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{\varepsilon}{m} F_{\mu\nu} \frac{dx^\nu}{d\tau} \tag{5}$$

The Hamiltonian corresponding to the Lagrangian (2) is found as

$$H = \frac{1}{2} g^{\mu\nu} (p_\mu - \varepsilon A_\mu)(p_\nu - \varepsilon A_\nu) \tag{6}$$

where $p_\mu = g_{\mu\nu} \dot{x}^\nu + \varepsilon A_\mu$ (7)

are the canonical momenta. Since H does not depend explicitly on l, the Hamiltonian is a constant of motion. As such, by using the normalization condition, one finds

$$H = -\frac{1}{2}m^2 \tag{8}$$

In the standard Boyer-Lindquist coordinates $(t,\ r,\ \theta,\ \varphi)$, the metric describing the Kerr-Newman de/anti de Sitter space-time takes the form,

$$\begin{aligned} ds^2 &= -\frac{\Delta_r}{\rho^2}[\frac{dt}{I} - \frac{a\sin^2\theta}{I}]^2 \\ &+ \frac{\Delta_\theta \sin^2\theta}{I^2}[\frac{a\,dt}{\rho} - \frac{(r^2+a^2)\,\mathrm{d}\phi}{\rho}]^2 \\ &+ \rho^2(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}) \end{aligned} \tag{9}$$

where the functions $\rho, \Delta_r, \Delta_\theta$ and I are defined respectively by

$$\left.\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta_r &= (1 - \frac{1}{3}\Lambda r^2)(r^2 + a^2) - 2Mr + e^2 \\ \Delta_\theta &= (1 + \frac{1}{3}\Lambda a^2\cos^2\theta) \\ I &= 1 + \frac{1}{3}\Lambda a^2 \end{aligned}\right\} \tag{10}$$

On the other hand, the electromagnetic field for the source is given by the required vector potential:

$$A_\mu = \frac{er}{I\rho^2}[\delta_\mu^t - a\sin^2\theta\delta_\mu^\phi] \tag{11}$$

The corresponding nonzero contravariant components $g^{\mu\nu}$ of the metric are obtained as

$$\left.\begin{aligned} g^{tt} &= \frac{I^2}{\Delta_r\Delta_\theta\rho^2}[\Delta_r a^2\sin^2\theta - \Delta_\theta(r^2+a^2)^2] \\ g^{\phi t} &= g^{t\phi} = \frac{I^2}{\Delta_r\Delta_\theta\rho^2}[\Delta_r - \Delta_\theta(r^2+a^2)^2] \\ g^{\phi\phi} &= I^2[\frac{1}{\sin^2\theta\Delta_\theta\rho^2} - \frac{a^2}{\Delta_r\rho^2}] \\ g^{rr} &= \frac{\Delta_r}{\rho^2} \\ g^{\theta\theta} &= \frac{\Delta_\theta}{\rho^2}. \end{aligned}\right\} \tag{12}$$

By equation (6), the general form of the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial \lambda} = -\frac{m^2}{2} = \frac{1}{2} g^{\mu\nu} [\frac{\partial S}{\partial x^i} - \varepsilon A_\mu][\frac{\partial S}{\partial x^\nu} - \varepsilon A_\nu] \quad (13)$$

whose solution takes the form,

$$S = -\frac{1}{2} m^2 \lambda - Et + L\phi + \int^r \frac{\sqrt{R(r)}}{\Delta_r} dr + \int^\theta \frac{\sqrt{\mathrm{V}(\theta)}}{\Delta_\theta} d\theta \quad (14)$$

where $R(r) = I^2[E(r^2+a^2) - aL + \frac{er\varepsilon}{I}]^2 - \Delta_r(Q + I^2(aE-L)^2 + m^2r^2)$ (15)

and $V(\theta) = (Q + I^2(aE-L)^2 - m^2a^2\cos^2\theta)\Delta_\theta - \frac{I^2}{\sin^2\theta}(aE\sin^2\theta - L)^2.$ (16)

Here Q is the Carter constant. Using the action given in equation (14), the following differential equations governing the motion of the test particle can be deduced:

$$\rho^2 \frac{\partial r}{\partial \lambda} = \pm\sqrt{R(r)} \quad (17)$$

$$\rho^2 \frac{\partial \theta}{\partial \lambda} = \pm\sqrt{V(\theta)} \quad (18)$$

$$\rho^2 \frac{\partial t}{\partial \lambda} = I^2(\mathrm{r}^2 + a^2)[E(r^2+a^2) - aL + \frac{er\varepsilon}{I}] - \frac{I^2 a}{\Delta_\theta}(\mathrm{a} E \sin^2\theta - \mathrm{L}) \quad (19)$$

$$\rho^2 \frac{\partial \phi}{\partial \lambda} = -\frac{I^2}{\Delta_\theta \sin^2\theta}(aE\sin^2\theta - L) + \frac{I^2 a}{\Delta_r}[E(r^2+a^2) - aL + \frac{er\varepsilon}{I}]. \quad (20)$$

where

$$\left.\begin{aligned}
\rho^2 &= r^2 + a^2\cos^2\theta \\
\Delta_r &= (1 - \frac{1}{3}\Lambda r^2)(r^2 + a^2) - 2Mr + e^2 \\
\Delta_\theta &= (1 + \frac{1}{3}\Lambda a^2\cos^2\theta) \\
I &= 1 + \frac{1}{3}\Lambda a^2
\end{aligned}\right\} \tag{21}$$

The functions $R(r)$ and $V(\theta)$ serve as the effective potentials defining the motion of the test particle in r- and *q*- directions [11]. Thus, the study of a test particle in the gravitational field of the Kerr-Newman de/anti de Sitter space-time is reduced to the study of motion of the test particle in the effective potentials $R(r)$ and $V(\theta)$.

For a circular orbit for which $\dot{r}=0$, the following conditions are satisfied at some radius r:

$$R(r) = 0, \tag{22}$$

$$R'(r) \equiv \frac{dR(r)}{dr} = 0. \tag{23}$$

It may also be verified that the condition $R''(r) < 0$ is satisfied showing that the circular orbit is stable.

Discussion on singularities and event horizons

The Kerr-Newman de/anti de Sitter metric (9) have singularities at r=0 and at $\Delta_r = 0$. The physically reasonable singularity is located at r=0 which is satisfied with q=p/2 and r=0. The condition $\Delta_r = 0$ implies that the Kerr-Newman anti de Sitter metric (9) exhibits four radial horizons. These radial horizons may be found as the roots of the equation $\Delta_r = 0$ which may be put in the form,

$$-\frac{\Lambda}{3}r^4 + (1 - \frac{a^2\Lambda}{3})r^2 - 2Mr + a^2 + e^2 = 0. \tag{24}$$

The four roots are:

$$r_{\pm}=\frac{\sqrt{X}}{2}\pm\frac{1}{2}\left[A-B+\frac{(2Z)^{\frac{1}{3}}}{3L(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}+\frac{(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}{3L2^{\frac{1}{3}}}-\frac{4M}{L\sqrt{X}}\right]^{\frac{1}{2}} \tag{25}$$

$$r_c=-\frac{\sqrt{X}}{2}-\frac{1}{2}\left[A-B+\frac{(2Z)^{\frac{1}{3}}}{3L(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}+\frac{(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}{3L2^{\frac{1}{3}}}+\frac{4M}{L\sqrt{X}}\right]^{\frac{1}{2}} \tag{26}$$

$$r_n=-\frac{\sqrt{X}}{2}+\frac{1}{2}\left[A-B+\frac{(2Z)^{\frac{1}{3}}}{3L(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}+\frac{(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}{3L2^{\frac{1}{3}}}+\frac{4M}{L\sqrt{X}}\right]^{\frac{1}{2}} \tag{27}$$

In which

$$L=\frac{\Lambda}{3},\quad A=\frac{(1-a^2L)}{3L},\quad B=\frac{(-1+a^2L)}{L},$$

$$Z=(1-14a^2L-12e^2L+a^4L^2)^3,$$

$$X=-A-B-\frac{(2Z)^{\frac{1}{3}}}{3L(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}-\frac{(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}{3L2^{\frac{1}{3}}}.,$$

Three out of the four roots of equation (24), have physical interpretations as follows: r_+ and r_- are the outer and inner event horizons of the black hole; r_c is the cosmological horizon for an observer between r_+ and r_c. Using the L. Ferrari's method [12], it may be shown that the real solutions for the horizon equation (24) are controlled by a factor h, called the horizon parameter, defined as

$$h\equiv\frac{1}{16\Lambda^3}\left[4(a^2+e^2)-\frac{1}{\Lambda}(1-\frac{\Lambda a^2}{3})^2\right]^3+$$
$$\frac{1}{16\Lambda^4}\left[(1-\frac{\Lambda a^2}{3})\{\frac{1}{\Lambda}(1-\frac{\Lambda a^2}{3})^2+12(a^2+e^2)\}-18M^2\right]^2. \tag{28}$$

For the negative cosmological constant $\Lambda<0$, some particular cases arise:

i. $h>0$: two real solutions, r_+ and r_- are expected;

ii. $h<0$: no real solution exists, thus providing a naked singularity;

iii. $h = 0$: r_+ and r_- coincide forming a single event horizon.

For the positive cosmological constant $\Lambda > 0$, depending on the value of h, there arise several physical interpretations:

i. $h < 0$: the roots r_-, r_+ and r_c are all real and positive showing that the black hole has well-defined horizons.

ii. $h = 0$: the event horizons, and become degenerate.

iii. $h > 0$, there exists only one horizon.

Geometrical surfaces of Kerr-Newman de /anti de Sitter space-time

In the Boyer-Lindquist coordinates, the stationary limit surfaces (SLS) of Kerr-Newman de/anti de Sitter black hole are obtainable by setting the roots

$g_{tt} \equiv \left[\dfrac{\Delta_\theta a^2 \sin^2\theta - \Delta_r}{I^2\rho^2}\right] = 0$. In the light of equation (10), these conditions give rise to a fourth order equation,

$$-\frac{\Lambda}{3}r^4 + r^2\left(1-\frac{3a^2}{\Lambda}\right) - 2Mr + \left(1+\frac{a^2\Lambda\cos^2\theta}{3}\right)a^2\sin^2\theta + (a^2+e^2) = 0. \quad (29)$$

taking $C = \left(1 + La^2cos^2\theta\right)a^2\sin^2\theta + a^2 + e^2$, the four roots of (29) are found as

$$r_{SE/I} = \frac{\sqrt{X}}{2} \pm \frac{1}{2}\left[A - B + \frac{(2Z)^{\frac{1}{3}}}{3L(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}} + \frac{(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}{3L2^{\frac{1}{3}}} - \frac{4M}{L\sqrt{X}}\right]^{\frac{1}{2}} \quad (30)$$

$$r_{SC} = -\frac{\sqrt{X}}{2} - \frac{1}{2}\left[A - B + \frac{(2Z)^{\frac{1}{3}}}{3L(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}} + \frac{(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}{3L2^{\frac{1}{3}}} + \frac{4M}{L\sqrt{X}}\right]^{\frac{1}{2}} \quad (31)$$

$$r_{S} = -\frac{\sqrt{X}}{2} + \frac{1}{2}\left[A - B + \frac{(2Z)^{\frac{1}{3}}}{3L(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}} + \frac{(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}{3L2^{\frac{1}{3}}} + \frac{4M}{L\sqrt{X}}\right]^{\frac{1}{2}} \quad (32)$$

$$X = -A - B - \frac{(2Z)^{\frac{1}{3}}}{3L(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}} - \frac{(Y+\sqrt{(Y^2-4Z)})^{\frac{1}{3}}}{3L2^{\frac{1}{3}}}.$$

For each radial horizon defined by (24) there is an associated stationary limit surface (SLS) defined by (29). Both the hyper surfaces given by equations (24) and (29) coincide at $\theta = 0,\ \pi$. The conditions for the existence of two distinct interior and exterior horizons for $\Lambda < 0$ and three real horizons , interior, exterior and cosmological horizons for $\Lambda > 0$ roots are respectively given by $X_{erg} > 0$ (for) and (for) where

$$X_{erg} \equiv \frac{1}{16\Lambda^3}\left[4(a^2+e^2)-\frac{1}{\Lambda}(1-\frac{\Lambda a^2}{3})^2-4a^2\sin^2\theta(1+\frac{\Lambda a^2\cos^2\theta}{3})\right]^3+$$
$$\frac{1}{16\Lambda^4}\left[(1-\frac{\Lambda a^2}{3})\{\frac{1}{\Lambda}(1-\frac{\Lambda a^2}{3})^2+12(a^2+e^2)-12a^2\sin^2\theta(1+\frac{\Lambda a^2\cos^2\theta}{3})\}-18M^2\right]^2. \tag{33}$$

Circular orbit of a test charged particle inside the inner horizon

We discuss the non rotating charged black hole: $a=0, m=1, Q=0$. With the help of equations (22) and (23), we obtain a pair of coupled equations for the energy E and angular momentum L of the test charged particle:

$$(Er^2+er\varepsilon)^2-(r^2-\frac{1}{3}\Lambda r^4-2r+e^2)(L^2+r^2)=0 \tag{34}$$

$$2(Er^2+er\varepsilon)(2Er+e\varepsilon)-(L^2+r^2)(-2+2r-\frac{4r^3\Lambda}{3})$$
$$-2r(-2r+r^2+e^2-\frac{\Lambda r^4}{3})=0. \tag{35}$$

As the roots of the equations (34) and (35) we obtain the following the pairs of values for E and L:

$$E_{1,2}=\pm\frac{\Delta D_1-e\varepsilon(-4r+r^2+3e^2+\frac{\Lambda r^4}{3})}{2r(-3r+r^2+2e^2)} \tag{36}$$

$$L^2_{1,2}=\frac{r^2}{(r^2-3r+2e^2)}[(r-e^2-\frac{\Lambda r^4}{3})+\frac{e\varepsilon\Delta(e\varepsilon\pm D_1)}{2(r^2-3r+2e^2)}] \tag{37}$$

where $\Delta = e^2-2r+r^2-\frac{\Lambda r^4}{3}$, $D_1^2=(8e^2+e^2\varepsilon^2-12r+4r^2)$.

The stability condition $R''(r) < 0$ for the non rotating circular orbit in the region of $0 < r < r_-$ is satisfied for (E_1, L_1) for all $\varepsilon > m$. The interaction between the charge of the particle and the electromagnetic field of the black hole may account for the existence of stable circular orbit of the charged particle inside the inner horizon [4].

Conclusion

We have shown that charged particles may have stable periodic orbits inside the inner horizon of Kerr-Newman de/anti de Sitter black hole. The interaction between the charge of the particle and the electromagnetic field of the black hole may account for existence of stable circular orbits in the case of a charged particle inside the inner horizon.

Reference

1. Chandrasekhar, S., The Mathematical Theory of Black Holes, Clarendon Press, Oxford (1983).
2. Dokuchaev, V. I., Is there any life inside black hole?, *Classical and Quantum Gravity*, **28**(23), 235015-235026 (2011); ArXiv: 1103.6140[gr-qc] (2010).
3. Bicak, J., Stuchlik, Z., and Balek, V., The motion of charged particles in the field of rotating charged black holes and naked singularities – I, *Bulletin of the Astronomical Institutes of Czechoslovakia*, **40**(2), 65-92 (1989).
4. Balek, V., Bicak, J., and Stuchlik, Z., The motion of the charged particles in the field of rotating charged black holes and naked singularities –II, *Bulletin of the Astronomical Institutes of Czechoslovakia*, **40**(3), 133-165 (1989).
5. Hackmann, E., Kagramanova, V., Kunz, J., and Lammerzahl, C., Analytical solution of the geodesic equation in Kerr-(anti)de Sitter space-times, *Physical Review D*, **81**, 044020-38 (2010); ArXiv: 1009.6117v1[gr-qc] (2010)
6. Sahni, V., and Starobinsky, A., The case for a positive cosmological Λ-term, *International Journal of Modern Physics D*, **9**(4), 373-443 (2000); Sahni, V. Energy density of relic gravity waves from inflation, *Physical Review D*, **42**(2), 453-463 (1990).
7. Perlmutter, S., et al., Discovery of a supernova explosion at half the age of the universe, *Nature*, **391**(6662), 51-54 (1998); Perlmutter, S., et al., Measurements of the cosmological parameters &! and Ë from the first seven supernovae at z e" 0.35, *Astrophysical Journal*, **483**(2), 565-581(1997); Riess, A. G., et al., Observational evidence from supernova for an accelerating universe and a cosmological constant, *Astronomical*

Journal, **116**(3),1009-1038 (1998); Perlmutter, S., Measurement of &! and Ë from 42 high-redshift supernovae, *Astrophysical Journal,* **517**(2),565-586(1999).

8. Sen, A., Developments in Superstring Theory, ArXiv: hep-th/9810356v2 (1998).
9. Carter, B., Global structure of the Kerr family of gravitational fields, *Physical Review*, **174**(1), 1559-1571 (1968).
10. Padmanabhan, T., Gravitation: Foundations and Frontier, Cambridge University Press, Cambridge (2010).
11. Bardeen, J. M, Press, W. H., and Teukolsky, S. A., Rotating black holes: locally nonrotating frames, energy extraction and scalar synchrotron radiation, *Astrophysical. Journal,* **178**(2), 347-369 (1972).
12. Cardano, Gerolamo (1545), ”Ars magna or The Rules of Algebra,” Dover (1993).

3

Theoretical Study of MHD Chemically Reacting and Radiating Fluid past an Oscillating vertical Plate embedded in a Darcian Porous Medium: An Exact Solution

Abdul Batin

Abstract

An analysis is carried out for the homogeneous chemical reaction effects on an unsteady magnetohydrodynamics free convection fluid flow past a semi-infinite isothermal vertical oscillating plate with variable mass diffusion embedded in a porous medium with thermal radiation. The fluid considered here is a gray, absorbing-emitting radiation but a non-scattering medium. The non-dimensional governing equations are formed with the help of suitable dimensionless governing parameter. The resultant coupled non dimensional governing equations are solved by ***Laplace Transform*** method, when the plate is oscillating harmonically in its own plane. The effect of important physical parameters on the velocity, temperature and concentration are shown graphically. It is found that the velocity increases with porosity parameter. Also it was seen that the velocity and concentration reduces with increasing chemical reaction parameter.

Key words: *Darcian drag force; Chemical reaction; Radiating Fluid; Isothermal vertical plate; Magnetic field.*

Introduction

Convective flow and heat transfer in a saturated porous medium has gained growing interest. This fact has been motivated by its importance in many engineering applications such as building thermal insulation, geothermal systems, food processing and grain storage, solar power collectors, contaminant transport in groundwater, casting in manufacturing processes, drying processes, nuclear waste, just to name a few. A theoretical and experimental work on this subject can be found in the recent monographs by Ingham and Pop [1] and Nield and Bejan [2]. Suction/blowing on convective heat transfer over a vertical permeable surface embedded in a porous medium was analyzed by Cheng [3]. In that work an application to warm water discharge along the well or fissure to an aquifer of infinite extent is discussed. Kim and Vafai [4] have analyzed the buoyancy driven flow about a vertical plate for constant wall temperature and heat flux. Raptis and Singh [5] studied flow past an impulsively started vertical plate in a porous medium by a finite difference method. The effect of radiation on free convection flow of fluid with variable viscosity from a porous plate is discussed Anwar and Hossain *et al* [6]. The fluid considered in that paper is an optically dense viscous incompressible fluid of linearly varying temperature dependent viscosity.

Soundalgekar and Ramana [7] have investigated the constant surface velocity case with a power-law temperature variation. Grubka and Bobba [8] have analyzed the stretching problem for a surface moving with a linear velocity and with a variable surface temperature. Radiation effect on mixed convection along a isothermal vertical plate were studied by Hossain and Thahar [9]. The governing equations were solved analytically. Das *et al.* [10] have analyzed radiation effects on flow past an impulsively started infinite isothermal vertical plate. The flow of a viscous, incompressible fluid past an infinite isothermal vertical plate, oscillating in its own plane, was solved by Soundalgekar [11]. The effect of mass transfer on the flow past an infinite vertical oscillating plate in the presence of constant heat flux has been studied by Soundalgekar *et al.* [12]. Ahmed and Kalita [13] presented the magnetohydrodynamic transient convective radiative heat transfer one-dimensional flow in an isotropic, homogenous porous regime adjacent to a hot vertical plate.

It is proposed to study the effects of porosity of the medium and chemical reaction on unsteady flow of electrically conducting laminar Newtonian fluid past an infinite isothermal vertical oscillating plate, in the presence of magnetic field and thermal radiation. The dimensionless governing equations are tackled using the ***Laplace Transform*** technique and the resultant solutions are in terms of exponential and complementary error function.

Mathematical Analysis

One dimensional unsteady flow of a viscous incompressible chemically reacting fluid which is initially at rest and surrounds an infinite vertical plate embedded in porous medium with temperature $\overline{T}_\infty$ and concentration $\overline{C}_\infty$. The x-axis is taken along the plate in the vertically upward direction and the y-axis is taken normal to the plate. Initially, it is assumed that the plate and the fluid are of the same temperature and concentration. At time $\overline{t} > 0$, the plate starts oscillating in its own plane with frequency $\overline{\omega}$ and the temperature of the plate is raised to $\overline{T}_w$ and the concentration level near the plate are also raised linearly with time ‘’. The plate is also subjected to a uniform magnetic field or strength B_0. The fluid considered here is a gray, absorbing emitting radiation but a non-scattering medium. It is assumed that the effect of viscous dissipation is negligible in the energy equation and there is a first order chemical reaction between the diffusing species and the fluid.

Under these assumptions, along with Boussinesq’s approximations, the boundary layer equations describing this flow as:

$$\frac{\partial \overline{u}}{\partial \overline{t}} = g\beta(\overline{T} - \overline{T}_\infty) + g\beta^*(\overline{C} - \overline{C}_\infty) + \nu \frac{\partial^2 \overline{u}}{\partial^2 \overline{y}} - \frac{\sigma B_0^2}{\rho}\overline{u} - \frac{\nu}{\overline{K}}\overline{u} \tag{1}$$

$$\rho C_P \frac{\partial \overline{T}}{\partial \overline{t}} = \kappa \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} - \frac{\partial q_r}{\partial \overline{y}} \tag{2}$$

$$\frac{\partial \overline{C}}{\partial \overline{t}} = D \frac{\partial^2 \overline{C}}{\partial \overline{y}^2} - \overline{C}_r^2 \overline{C} \tag{3}$$

In most cases of chemical reactions, the rate of reaction depends on the concentration of the species itself. A reaction is said to be of the order n, if the reaction rate is proportional of the n^{th} power of the concentration. In particular, a reaction is said to be first order, if the rate of reaction is directly proportional to concentration itself. The initial and boundary conditions:

$$\bar{t} \le 0 : \bar{u} = 0, \bar{T} = \bar{T}_\infty, \bar{C} = \bar{C}_\infty \text{ for all } \bar{y}$$

$$\bar{t} > 0 : \bar{u} = u_0 \cos \bar{\omega}\bar{t}, \bar{T} = \bar{T}_w, \bar{C} = \bar{C}_w + (\bar{C}_w - \bar{C}_\infty) A\bar{t} \text{ at } \bar{y} = 0$$

$$\bar{t} > 0 : \bar{u} \to 0, \bar{T} \to \bar{T}_\infty, \bar{C} = \bar{C}_\infty \text{ as } \bar{y} \to \infty \tag{4}$$

where , $A = \dfrac{u_0^{\,2}}{\nu}$

The local radiant for the case of an optically thin gray gas is defined as

$$\frac{\partial q_r}{\partial \bar{y}} = -4a^* \sigma (\bar{T}_\infty^4 - \bar{T}^4) \tag{5}$$

It is assume that the temperature differences within the flow are sufficiently small such that $\bar{T}^4$ may be expressed as a linear function of the temperature. This is accomplished by expanding $\bar{T}^4$ in a taylor series about about $\bar{T}_\infty$ and neglecting higher-order terms, thus

$$\bar{T}^4 \cong 4\bar{T}_\infty^3 \bar{T} - 3\bar{T}_\infty^4 \tag{6}$$

On using equations (5) and (6), equation (2) reduces to

$$\rho C_P \frac{\partial \bar{T}}{\partial \bar{t}} = \kappa \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + 16 a^* \sigma \bar{T}_\infty^3 (\bar{T}_\infty - \bar{T}) \tag{7}$$

The following non-dimensional quantities are:

$$u = \frac{\bar{u}}{u_0}, t = \frac{\bar{t} u_0^2}{\nu}, y = \frac{\bar{y} u_0}{\nu}, \theta = \frac{\bar{T} - \bar{T}_\infty}{\bar{T}_w - \bar{T}_\infty}, K = \frac{u_0^2 \bar{K}}{\nu^2}, Sc = \frac{\nu}{D},$$

$$Gr = \frac{g\beta\nu(\bar{T}_w - \bar{T}_\infty)}{u_0^3}, \phi = \frac{\bar{C} - \bar{C}_\infty}{\bar{C}_w - \bar{C}_\infty}, Gc = \frac{\nu g \beta^* (\bar{C}_w - \bar{C}_\infty)}{u_0^3}, \tag{8}$$

$$\omega = \frac{\overline{\omega}\nu}{u_0^2}, R_a = \frac{16a^*\nu^2\sigma\overline{T}_\infty^3}{\kappa u_0^2}, \Pr = \frac{\mu C_p}{\kappa}, M = \frac{\sigma B_0^2 \nu}{\rho u_0^2}, C_r^2 = \frac{\nu \overline{C}_r^2}{u_0^2}$$

The transformed equations of (1), (3) and (7) are

$$\frac{\partial u}{\partial t} = Gr\theta + Gc\phi + \frac{\partial^2 u}{\partial y^2} - (M + K^{-1})u \tag{9}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2} - \frac{R}{Pr}\theta \tag{10}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2} - \frac{R}{\Pr}\theta \tag{11}$$

The initial and boundary conditions in non-dimensional form are

$u = 0,\ \theta = 0,\ \phi = 0$ for all $t,\ y \le 0$

$t > 0: u = \cos\omega t, \theta = 1, \phi = t$ at $y = 0$

$$t > 0: u \to 0,\ \theta \to 0,\ \phi \to 0 \text{ as } y \to \infty \tag{12}$$

By the usual Laplace-transform technique, the exact solutions of the equations (9) to (11), subject to the boundary conditions (12), are:

$$\theta = \frac{1}{2}\left[\begin{array}{l}\exp(2\eta\sqrt{Rt})erfc(\eta\sqrt{Pr} + \sqrt{at}) \\ + \exp(-2\eta\sqrt{Rt})erfc(\eta\sqrt{Pr} - \sqrt{at})\end{array}\right] \tag{13}$$

$$\phi = \frac{t}{2}\left[\begin{array}{l}\exp(2\eta\sqrt{KtSc})erfc(\eta\sqrt{Sc} + \sqrt{Kt}) \\ + \exp(-2\eta\sqrt{KtSc})(\eta\sqrt{Sc} - \sqrt{Kt})\end{array}\right]$$

$$-\frac{\eta\sqrt{Sct}}{2\sqrt{K}}\left[\begin{array}{l}\exp(-2\eta\sqrt{KtSc})erfc(\eta\sqrt{Sc} - \sqrt{Kt}) \\ - \exp(2\eta\sqrt{KtSc})erfc(\sqrt{Sc} + \sqrt{Kt})\end{array}\right] \tag{14}$$

$$
\begin{aligned}
u = \exp\frac{(\mathrm{i}\omega \mathrm{t})}{4} & \begin{bmatrix} \exp(2\eta\sqrt{(N+i\omega)t} \ erfc(\eta+\sqrt{(N+i\omega)t}) \\ +\exp(-2\omega\sqrt{(N+i\omega)t}) \ erfc(\eta-\sqrt{(N+i\omega)t}) \end{bmatrix} \\
+\frac{\exp(-i\omega t)}{4} & \begin{bmatrix} \exp(2\eta\sqrt{(N-i\omega)t}) \ erfc(\eta+\sqrt{(N-i\omega)t}) \\ +\exp(-2\eta\sqrt{(N-i\omega)t} \ erfc(\eta-\sqrt{(N-i\omega)t}) \end{bmatrix} \\
+\left(d+P_5(1+P_3t)\right) & \begin{bmatrix} \exp\left(2\eta\sqrt{Nt}\right) erfc\left(\eta+\sqrt{Nt}\right) \\ +\exp\left(-2\eta\sqrt{Nt}\right) erfc\left(\eta-\sqrt{Nt}\right) \end{bmatrix} \\
-\frac{P_5P_3\eta\sqrt{t}}{\sqrt{N}} & \begin{bmatrix} \exp\left(-2\eta\sqrt{Nt}\right) erfc\left(\eta-\sqrt{Nt}\right) \\ -\exp\left(2\eta\sqrt{Nt}\right) erfc\left(\eta+\sqrt{Nt}\right) \end{bmatrix} \\
-P_4\exp(bt) & \begin{bmatrix} \exp\left(-2\eta\sqrt{(N+P_2)t}\right) erfc\left(\eta-\sqrt{(N+P_2)t}\right) \\ +\exp\left(2\eta\sqrt{(N+P_2)t}\right) erfc\left(\eta+\sqrt{(N+P_2)t}\right) \end{bmatrix} \\
-P_5\exp(P_3t) & \begin{bmatrix} \exp\left(-2\eta\sqrt{(M+P_3)t}\right) erfc\left(\eta-\sqrt{(M+P_3)t}\right) \\ +\exp\left(2\eta\sqrt{(M+P_3)t}\right) erfc\left(\eta+\sqrt{(M+P_3)t}\right) \end{bmatrix} \\
-d & \begin{bmatrix} \exp\left(-2\eta\sqrt{R_at}\right) erfc\left(\eta\sqrt{Pr}+\sqrt{P_1t}\right) \\ +\exp\left(-2\eta\sqrt{R_at}\right) erfc\left(\eta\sqrt{Pr}-\sqrt{P_1t}\right) \end{bmatrix} \\
+P_4\exp(bt) & \begin{bmatrix} \exp\left(-2\eta\sqrt{Pr(P_1+P_2)t}\right) erfc\left(\eta\sqrt{Pr}-\sqrt{(P_1+P_2)t}\right) \\ +\exp\left(2\eta\sqrt{Pr(P_1+P_2)t}\right) erfc\left(\eta\sqrt{Pr}+\sqrt{(P_1+P_2)t}\right) \end{bmatrix} \\
-P_4(1+ct) & \begin{bmatrix} \exp\left(2\eta\sqrt{KtSc}\right) erfc\left(\eta\sqrt{Sc}+\sqrt{Kt}\right) \\ +\exp\left(-2\eta\sqrt{KtSc}\right) erfc\left(\eta\sqrt{Sc}-\sqrt{Kt}\right) \end{bmatrix} \\
+\frac{P_3P_5\eta\sqrt{Sct}}{\sqrt{M}} & \begin{bmatrix} \exp\left(-2\eta\sqrt{KtSc}\right) erfc\left(\eta\sqrt{Sc}-\sqrt{Kt}\right) \\ -\exp\left(2\eta\sqrt{KtSc}\right) erfc\left(\eta\sqrt{Sc}+\sqrt{kt}\right) \end{bmatrix}
\end{aligned}
$$

$$+ P_5 \exp(ct) \begin{bmatrix} \exp\left(-2\eta\sqrt{Sc(K+P_3)t}\right) erfc\left(\eta-\sqrt{Sc}-\sqrt{(K+P_3)t}\right) \\ +\exp\left(2\eta\sqrt{Sc(K+P_3)t}\right) erfc\left(\eta\sqrt{Sc}+\sqrt{K+P_3)t}\right) \end{bmatrix}$$

Where,

$$N = M + K^{-1},\ P_1 = \frac{R}{\Pr},\ P_2 = \frac{R}{1-\Pr},\ P_3 = \frac{M-KSc}{1-Sc},$$

$$P_4 = \frac{Gr}{2P_2(1-\Pr)},\ P_5 = \frac{Gc}{2P_3^2(1-Sc)},\ \eta = y/2\sqrt{t}$$

Results and Discussions

The purpose of the calculations given here is to assess the effects of the parameters ***M*, *K*, C$_r$,** and ***R$_a$*** upon the nature of the flow and transport. All the numerical calculations are done through out the discussions with respect to flow parameters M = 5, ωt = π/3, K=0.5, R_a=2, C_r= 0.3 and Sc = 0.78. The value of Prandtl number Pr is chosen such that they represent water **(*Pr* = 7.0)**.

The effect of magnetic field parameter M on the velocity is shown in Fig. 3.1. The velocity decreases with an increase in the magnetic parameter. It is because that the application of transverse magnetic field will result a resistive type force (***Lorentz force***) similar to drag force which tends to resist the fluid flow and thus reducing its velocity. Also, the boundary layer thickness decreases with an increase in the magnetic parameter.

The variations of the thermal radiation parameter Ra on the velocity and temperature are shown in Figs. 3.2 and 3.3 respectively. The radiation parameter Ra defines the relative contribution of conduction heat transfer to thermal radiation transfer. It is obvious that an increase in the radiation parameter results in decreasing velocity and temperature within the boundary layer.

Figure 3.4 shows the effect of the permeability of the porous medium parameter K on the velocity distribution. It is found that the velocity increases with an increase in K.

Figures 3.5 and 3.6, illustrates the behavior velocity and concentration for different values of chemical reaction parameter C_r . It is observed that an increase in leads to a decrease in both the values of velocity and concentration.

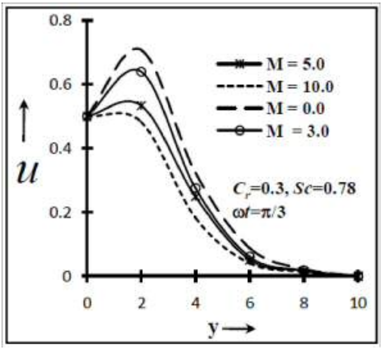

Fig.3.1Velocity profile for M

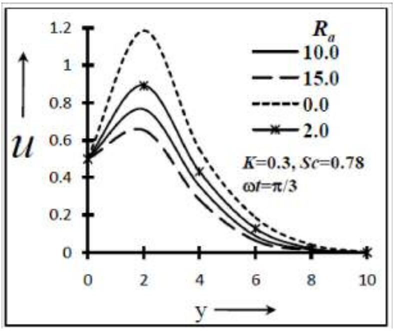

Fig.3.2 Velocity profile for Ra

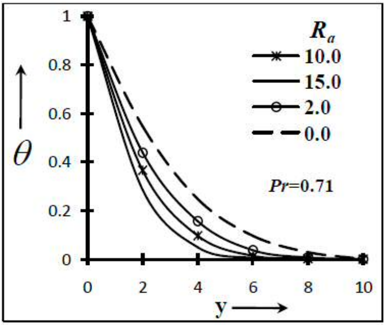

Fig.3.3 Temperature profile for Ra

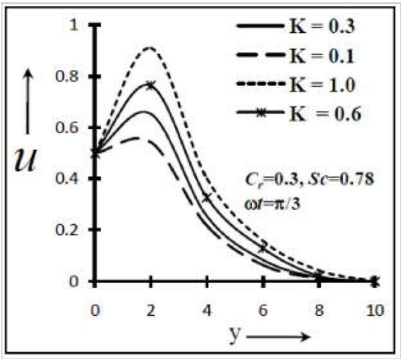

Fig.3.4Velocity profile for K

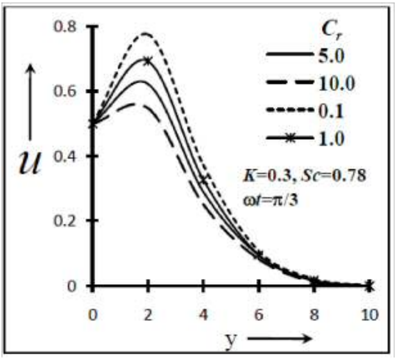

Fig. 3.5 Velocity profile for Cr

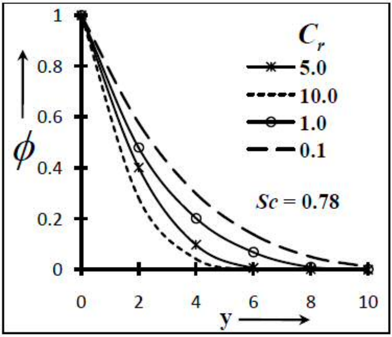

Fig. 3.6 Concentration profile forCr

Conclusions

In this paper, an unsteady one dimensional thermal radiations and MHD free convection flow past a moving vertical plate with first order chemical reaction was considered. The non-dimensional governing equations are solved with the help of Laplace transform technique. The conclusions of the study are as follows:

1. With an increase in the magnetic parameter the flow velocity decelerated.
2. The velocity increases with an increase in the permeability of the porous medium parameter.
3. An increase in the thermal radiation leads to decreases both the flow velocity and temperature.
4. The velocity as well as concentration decreases with an increase in the chemical reaction parameter.

Nomenclature

A	Constant.
a^*	Absorption Coefficient.
B_0	External Magnetic Field.
$\overline{C}$	Species Concentration in the Fluid.
f	Dimensionless Concentration.
C_p	Specific heat at constant pressure.
D	Mass Diffusion Coefficient.
Gr	Mass Grashof Number.
Gc	Thermal Grashof Number.
g	Acceleration due to gravity.
$\overline{C}_r$	Chemical Reaction Parameter.
C_r	Dimensionless Chemical Reaction Parameter
$\overline{K}$	Porosity parameter
K	Dimensionless porosity parameter
M	Magnetic Field Parameter.
Pr	Prandtl Number
R_a	Radiation Parameter.

Sc Schmidt Number.

$\overline{T}$ Temperature of the fluid near the plate.

$\bar{t}$ Time.

t Dimensionless time

$\overline{u}$ Velocity of the fluid in the $\overline{x}$ direction.

u_0 Velocity of the plate.

u Dimensionless velocity.

$\overline{y}$ Coordinate axis normal to the plate.

y Dimensionless coordinate axis normal to the plate.

Greek Symbols:

α Thermal Diffusivity.

β Volumetric Coefficient of thermal Expansion.

β^* Volumetric coefficient of expansion with concentration.

μ Cofficient of Viscosity.

ν Kinematic Viscosity

k Thermal Conductivity.

ωt Phase angle

ρ Density of the Fluid.

θ Dimmensionless Temperature.

η Similarity Parameter.

erfc Complementary Error Function.

Subscripts:

w Conditions at the wall.

∞ Conditions in the free stream.

References

1. Ingham, D. B., and I. Pop (Eds.) (1998). *Transport Phenomena in Porous Media*, Pergamon, Oxford. 1998.
2. Nield. D. A. and Bejan, A. (1998). *Convection in Porous Media,* 2. Ed., Springer-Verlag, Berlin.

3. Cheng, P. (1977). The influence of lateral mass flux on free convection boundary layers in a saturated porous medium, *Int.J. Heat Mass Transfer*, 20, 201- 206.

4. Kim, S. and Vafai, K., (1989). Analysis of natural convection about a vertical plate embedded in a porous medium. *Int. J. Heat Mass Transfer*, 32, 665-677.

5. Raptis. A and Singh. A. K., (1985). Free convection flow past an impulsively started vertical plate in a porous medium by finite difference method. *Astrophys. Space Sci.*, 112, 259-265.

6. Anwar Hossain, M., Khanafer, K., Vafai, K., (2001). The effect of radiation on free convection flow of fluid with variable viscosity from a porous vertical plate, *Int. J. Thermal Sci,,* 40, pp. 115-124.

7. Soundalgekar, V. M., Ramana, T.V., (1980). Heat transfer past a continuous moving plate with variable temperature, *Warme- Und Stoffuber tragung*, 14, 91-93.

8. Grubka, L. J., Bobba, K. M. (1985). Heat transfer characteristics of a continuous stretching surface with variable temperature, *J Heat Transfer*, 107, 248-250.

9. Hossain, M.A., and H.S. Takhar, H.S., 1996. Radiation effect on mixed convection along a vertical plate with uniform surface temperature, Heat and Mass Transfer, Vol. 31, pp. 243-248.

10. Das U.N., Deka R.K., Soundalgekar V.M., 1996. Radiation effects on flow past an impulsively started vertical infinite plate, Journal of Theoretical Mechanics, Vol. 1, pp. 111-115.

11. Soundalgekar, V.M., 1979. Free Convection Effects on the Flow Past a Vertical Oscillating Plate, Astrophysics Space Science, Vol. 64, pp. 165-172.

12. Soundalgekar V.M., Lahurikar R.M., Pohanerkar S.G., and Birajdar N.S., 1994. Effects Mass Transfer on the Flow Past an Oscillating Infinite Vertical Plate with Constant Heat Fluk, Thermophysics and Aeromechanics, Vol. 1, pp. 119-124.

13. Ahmed, S. and Kalita, K., Magnetohydrodynamic transient flow through a porous medium bounded by a hot vertical plate in presence of radiation: A theoretical analysis, *J. Engineering Physics and Thermophysics*, **86**(1), 31-39 (2013).

4

Cone Metric Space and Common Fixed Point Theorem

Manoj Solanki

Abstract

The aim of this paper is to prove some common fixed point theorem in cone metric space for rational expression under normal cone setting. Our result generalize the main result of Jaggi.

Key Words: *Cone metric space common fixed point, metric space, normal cone, rational expression.*

AMS Classification: 47H10, 54H25, 55M20

Introduction

Fixed point theory plays a basic role in application of various branches of mathematics from elementary calculus and linear algebra to topology and analysis. Fixed point theory is not restricted to mathematics and this theory has many applications in other discipline. The Banach Contraction principle with rational expressions have been expanded and some common fixed point theorem have been obtained in [1] [2]. Cone metric space where consider by Huang and Zhang [4] who reintroduced the concept which has been known since the middle of 20th century.

They have considered convergent in cone metric space, introduced completeness of cone metric space and prove a Banach contraction mapping theorem and some other fixed point theorems involving contractive type mapping in cone metric space using the normality condition. Our results generalized the main result of Jaggi [3] with adding new mappings.

Preliminary

Let G be a real Banach space and $'K'$ a subset at G. K is called a cone iff

i. K is closed, nonempty, and $K \neq \{0\}$

ii. $a, b \in R, a, b \geq 0, \; x, y \in K \Rightarrow ax + by \in K.$

$x \in K \; and \; -x \in K \Rightarrow x = 0 \; i.e. \; K \cap (-K) = \{0\}.$

Given a cone $K \subset G,$ we define a partial ordering $\leq$ with respect to $K \; by \; x \leq y \; iff \; y - x \in K$

We write $x < y \; if \; x \leq y \; but \; x \neq y.$

While $x \ll y \; if \; y - x \in int \, K.$

The cone K is called normal if there is a number $M > 0 \; s.t. x, y \in G.$

$0 \leq x < y \; implies \; \| x \| \leq M \| y \| .$

The least positive number satisfying above is called the normal constant of K.

Definition: 2.1 [2]

Let x be non-empty set G is a real Banach space and $K \subset G,$ a cone. Suppose the mapping $d: X \times X \rightarrow G$ satifies

$d\,1.$ $0 < d\,(x, y) for \; all \; x, y \in X$

$and \; d\,(x, y) = 0 \; iff \; x = y$

$d\,2.$ $d\,(x, y) = d\,(y, x) for \; all \; x, y \in X$

$d\,3.$ $d\,(x, y) \leq d(x, z) + d(z, y) for \; all \; x, y, z \in X$

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2 [4]

Let $(X, d)$is said to be a complete cone metric space, if every Cauchy sequence is convergent in X.

Definition 2.3 [3]

Let (*X*, *d*) a cone metric space a self mapping *T* on *X* is called an jaggi contraction, if it satisfies the condition

$$d(Tx,Ty) \le \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta\, d(x,y) + L \min\{d(x,Ty), d(y,Tx)\}$$
$\forall\, x, y \in X,\ Where\ \ L \ge 0\ and\ \alpha, \beta \in [0,1)\ with\ \alpha + \beta < 1.$

Main Result

Theorem 3.1 Let (*X*, *d*) be a complete cone metric space *K* be normal cone with normal constant *M*.

Let $S, T: X \to X$.

$$d(Sx,Ty) \le \alpha \frac{d(x,Sx)d(y,Ty)}{d(x,y)} + \beta \frac{d(x,Sx)d(y,Ty) + d(x,Ty)d(y,Sx)}{d(x,y)}$$

$$+\gamma\, d(x,y) + L \min\{d(x,Ty), d(y,Sx)\}$$

For all $x, y \in X$ $where\ L \ge 0\ and\ \ \alpha, \beta, \gamma \in [0,1)\ with\ \alpha + 2\beta + 2\gamma < 1.$

Then *T* has a unique fixed point in *X*.

(3.1a)

Proof:

Choose $x_1 \in Sx_0\ and\ x_2 \in Tx_1\ \ s.t.\ x_{2n+1} = Sx_{2n}\ and\ x_{2n+2} = Tx_{2n+1}.$

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \le \alpha \frac{d(x_{2n}, Sx_{2n})d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1})}$$

$$+\beta \frac{d(x_{2n}, Tx_{2n})d(x_{2n+1}, Tx_{2n+1}) + d(x_{2n}, Tx_{2n+1})d(x_{2n+1}, Sx_{2n})}{d(x_{2n}, x_{2n+1})}$$

$$+\gamma\, d(x_{2n}, x_{2n+1}) + L \min\{d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n})\}$$

$$\le \alpha \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})}$$

$$+\beta \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}) + d(x_{2n}, x_{2n+2})d(x_{2n+1}, x_{2n+1})}{d(x_{2n}, x_{2n+1})}$$

$$+\gamma\, d\,(x_{2n},x_{2n+1}) + L\min\{d(x_{2n},x_{2n+2}),d(x_{2n+1},x_{2n+1})\}$$

$$=\alpha\, d(x_{2n+1}\,,x_{2n+2}) + \beta\, d(x_{2n+1},x_{2n+2}) + \gamma\, d\,(x_{2n},x_{2n+1})$$

$$d(x_{2n+1}\,,x_{n+2}) \le (\alpha+\beta)\, d\,(x_{2n+1},x_{2n+2}) + \gamma\, d\,(x_{2n},x_{2n+1})$$

$$d(x_{2n+1}\,,x_{2n+2}) \le \frac{\gamma}{1-\alpha-\beta}\, d\,(x_{2n},x_{2n+1})$$

$$d(x_{2n+1}\,,x_{2n+2}) \le s\, d\,(x_{2n},x_{2n+1})$$

where $s = \dfrac{\gamma}{1-\alpha-\beta}\,, \alpha+2\beta+2\gamma<1, 0<K<1$

Again by induction

$$d(x_{2n+3}\,,x_{2n+2}) = d\,(Sx_{2n+2}\,,Tx_{2n+1}) \le$$

$$\alpha\,\frac{d(x_{2n+2},Sx_{2n+2})d(x_{2n+1},Tx_{2n+1})}{d(x_{2n+2},x_{2n+1})}$$

$$+\beta\frac{d(x_{2n+2},Sx_{2n+2})d(x_{2n+1},Tx_{2n+1}) + d(x_{2n+2},Tx_{2n+1})d(x_{2n+1},Sx_{2n+2})}{d(x_{2n+2},x_{2n+1})}$$

$$+\gamma\, d\,(x_{2n+2},x_{2n+1}) + L\min\{d(x_{2n+2},Tx_{2n+1}),d(x_{2n+1},Sx_{2n+2})\}$$

$$\le \alpha\,\frac{d(x_{2n+2},x_{2n+3})d(x_{2n+1},x_{2n+2})}{d(x_{2n+2},x_{2n+1})}$$

$$+\beta\frac{d(x_{2n+2},x_{2n+3})d(x_{2n+1},x_{2n+2}) + d(x_{2n+2},x_{2n+2})d(x_{2n+1},x_{2n+3})}{d(x_{2n+2},x_{2n+1})}$$

$$+\gamma\, d\,(x_{2n+2},x_{2n+1}) + L\min\{d(x_{2n+2},x_{2n+2}),d(x_{2n+1},x_{2n+3})\}$$

$$\le \alpha\, d(x_{2n+2},x_{2n+3}) + \beta d(x_{2n+2},x_{2n+3}) + \gamma\, d\,(x_{2n+2},x_{2n+1})$$

$$d(x_{2n+3},x_{2n+2}) \le \frac{\gamma}{1-\alpha-\beta}\, d\,(x_{2n+2},x_{2n+1})$$

$$d(x_{2n+3},x_{2n+2}) \le s\, d\,(x_{2n+2},x_{2n+1})$$

where $s = \dfrac{\gamma}{1-\alpha-\beta}$, $\alpha + 2\beta + 2\gamma < 1, 0 < K < 1$ and by induction

$$d(x_{n+1}, x_{n+2}) \leq s\, d(x_n, x_{n+1}) \leq \cdots \leq s^n\, d(x_0, x_1)$$

By Triangle in equality

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots \ldots + d(x_{m-1}, x_m)$$

$$\leq (s^n + s^{n+1} + s^{n+2} + \ldots \ldots + s^{n+m-1}) d(x_0, x_1)$$

$$\leq \frac{s^n}{1-s} d(x_0, x_1)$$

We get

$$\| d(x_n, x_{n+1}) \| \leq M \frac{s^n}{1-s} \| d(x_0, x_1) \|$$

$\Rightarrow d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence, so by completeness of X this sequence must be convergent in X. Now we shall prove that w is anther common fixed point of S and T in X.

$$d(w, Tw) \leq d(w, x_{2n+1}) + d(x_{2n+1,} Tw)$$

$$\leq d(w, x_{n+1}) + d(Sx_{2n,} Tw) \leq d(w, x_{2n+1}) + \alpha \frac{d(x_{2n}, Sx_{2n}) d(w, Tw)}{d(x_{2n}, w)}$$

$$+\beta \frac{d(x_{2n}, Sx_{2n}) d(w, Tw) + d(x_{2n}, Tw) d(w, Sx_{2n})}{d(x_{2n}, w)} + \gamma\, d(x_{2n}, w)$$

$$L \min\{d(x_{2n}, Tw), d(w, Sx_{2n})\} \leq d(w, x_{2n+1}) + \alpha \frac{d(x_{2n}, x_{2n+1}) d(w, w)}{d(x_{2n}, w)}$$

$$+\beta \frac{d(x_{2n}, x_{2n+1}) d(w, w) + d(x_{2n}, w) d(w, x_{2n+1})}{d(x_{2n}, w)} + \gamma\, d(x_{2n}, w)$$

$$L \min\{d(x_{2n}, w), d(w, x_{2n+1})\}$$

$$\leq d(w, x_{2n+1}) + \beta\, d(w, x_{2n+1}) + \gamma\, d(x_{2n}, w)$$

$$+ L \min\{d(x_{2n}, w), d(w, x_{2n+1})\}$$

So using the condition of normality of cone

$$\| d(w,Tw) \| \leq M \, [\| d(w,x_{2n+1}) \| + \beta \; \| d(w,x_{2n+1}) \| + \gamma \, \| \; d\,(x_{2n},w) \|$$

$$+ \, L \min\{\| \, d(x_{2n},w), d(w,x_{2n+1}) \, \|\}$$

As $n \to 0$ we have

$\| \, d\,(w,Tw) \, \| \leq 0,$ Hence $w = Tw,$ w is a fixed point of T.

Similarly we have

$$d(w,Sw) \leq d(w,x_{2n+1}) + \; d(x_{2n+1,} \, Sw)$$

$$\leq d(w,x_{n+1}) + \; d(\,Sw,Tx_{2n+1})$$

$$\leq d(w,x_{2n+1}) + \alpha \, \frac{d(w,Sw)d(x_{2n+1},Tx_{2n+1})}{d(w, \;\; x_{2n+1})}$$

$$+\beta \, \frac{d(w,Sw)d(x_{2n+1},Tx_{2n+1}) + \; d(w,Tx_{2n+1})d(x_{2n+1},Sw)}{d(w, \;\; x_{2n+1})}$$

$$+\gamma \, d\,(w,x_{2n+1}) + \; L \min\{d(w,x_{2n+1}), d(x_{2n+1},Sw)\}$$

$$\leq d(w,x_{2n+1}) + \beta \; d(w,x_{2n+2}) + \gamma \; d\,(w,x_{2n+1})$$

$$+ \, L \min\{d(w,x_{2n+2}), d(x_{2n+1},w)\}$$

So using the condition of normality of cone

$$\| \, d(w,Sw) \, \| \leq M \, [\| \, d(w,x_{2n+1}) \, \| + \beta \; \| \, d(w,x_{2n+2}) \, \|$$

$$+\gamma \, \| \;\; d\,(w,x_{2n+1}) \, \| + L \min\{\| \, d(w,x_{2n+2}), d(\,x_{2n+1},w) \, \|\}]$$

As $n \to 0$ we have

$\| \, d\,(w,Sw) \, \| \leq 0,$ Hence $w = Sw,$ w is a fixed point of S.

Therefore w is a fixed point of S and T in X

Conclusion

In this paper, I proved common fixed point theorem for cone metric space which shows that our previous result Solanki Manoj et.al [13] is generalized version and some known results.

References

1. Fisher, B., and Khan, M. S., fixed point, common fixed point and constant mapping, sfudia sci. math. Hungar. II, 467-470 (1978).

2. Fisher, B., common fixed points and constant mappings rational inequality math. sem. notes (Univ Kobe) 1978.

3. Jaggi, D.S., Some unique fixed point theorems, Indian J. Pure appl. math., **8**, 223-230 (1977).

4. Huang L.G., and Zhang, X., cone metric space and fixed point theorems of contractive mapping, Jar of mathematical analysis and applications, **332** (2), 1468-1476 (2007).

5. Jungek G., and Rhoades B. E., fixed point for set valued function without continuities, *Indian J. pure Appl. math,* **29** (3), 227-238 (1998).

6. Pant, R. P., Common fixed points of contractive maps, *J. math and appl*., **226**, 251-258 (1998).

7. Abbas, M., and Rhoades B. E., fixed point result in cone metric space, *App. Mathematics Letter*, **22**(4), 511-515 (2009).

8. Kadelburg Z., Radenovils and Rakocevic V., Remarks on Quasi-contractions on a cone metric space, *App. Mathematics letter* , **22**, 1674- 1679 (2009).

9. Muhammad Arshad, and Karapinar E. Ahmad J., some unique fixed point theorems for rational contraction in partially ordered metric space, *Journal of Inequalities and Application*, 2013: 248 (2013)doi : 10. 1186 / 1029-242X-2013-248.

10. Jaggi, D. S. and Das B. K., An extension of Banach fixed point theorem through ratioanl expression, *Bull. of Math. Soc*., **72** , 261-262 (1980).

11. Das, B. K. and Gupta S., An extension of Banach contraction principle through rational expression, *Ind. J. of pure Applied Math*., **6**, 1455-1458 (1975).

12. Uthaykumar, R., and Prabhakar, G. A., Common fixed point theorem is cone metric space for rational contraction, *Int. J. of analysis and application*, **3**, 112-118 (2013).

13. Solanki, M., Bohara, A., and Bhardwaj R., Some common fixed point theorem under rational expressions in cone metric space, Conference proceedings, APCMET-2014, JNU, New-Delhi, 218-227, ISBN : 978 - 93 - 83083 - 71 - 8.

5

Analytical Solution for Steady Magnetohydrodynamic mixed Convection Transport in a Porous Media with Thermal Radiation and Ohmic Heating

Hamida Khatun

Abstract

Analytical solutions for the steady magnetohydrodynamic laminar mixed convection heat and mass transfer flow of viscous electrically conducting fluid past a vertical permeable surface embedded in a *Darcian* porous medium with thermal radiation and chemical reaction effects is presented. The heat equation includes the terms involving the radiative heat flux, Ohmic dissipation, viscous dissipation and the internal absorption whereas the mass transfer equation includes the effects of chemically reactive species of first-order. The non-linear coupled differential equations are solved analytically by perturbation technique. Validation of the analysis has been performed by comparing the present results with those available in the open literature [13] and a very good agreement has been established. It is observed that the effect of heat absorption is to decrease the velocity and temperature profiles in the boundary layer.

Keywords: *Ohmic heating; Viscous dissipation; Porosity; Mixed Convection; Heat absorption.*

Introduction

The effects of transversely applied magnetic field, on the flow of an electrically conducting fluid past an impulsively started infinite isothermal vertical plate studied by Soundalgekar et al. [2]. MHD effects on impulsively started vertical plate with variable temperature in the presence of transverse magnetic field were considered by Soundalgekar et al. [3]. Free convection flows in a porous media with chemical reaction have wide applications in geothermal and oil reservoir engineering as well as in chemical reactors of porous structure. Many transport processes exist in industrial applications in which the simultaneous heat and mass transfer occur as a result of combined buoyancy effects of diffusion of chemical species. Moreover, considerable interest has been evinced in radiation interaction with convection and chemical reaction for heat and mass transfer in fluids. This is due to the significant role of thermal radiation in the surface heat transfer when convection heat transfer is small, particularly, in free convection problems involving absorbing–emitting fluids. Khair and Bejan [3] studied heat and mass on flows past an isothermal flat plate. Lin and Wu [4] analyzed combined heat and mass transfer by laminar natural convection from a vertical plate. Yin [5] studied numerically the force convection effect on magnetohydrodynamics heat and mass transfer of a continuously moving permeable surface. Acharya *et al*. [6] have studied heat and mass transfer over an accelerating surface with heat source in the presence of suction and blowing. Muthucumaraswamy and Janakiraman [7] studied MHD and radiation effects on moving isothermal vertical plate with variable mass diffusion. Hossain *et al*. [8] investigated radiation effects on the free convection flow of an optically incompressible fluid along a uniformly heated vertical infinite plate with a constant suction. Orhan and Kaya [9] examined MHD mixed convective heat transfer along a permeable vertical infinite plate in the presence of radiation and solutions are derived using Kellar box scheme and accurate finite-difference scheme. Ahmed and Liu [10] examined the effects of mass transfer on a mixed convection three dimensional heat transfer flow of a viscous incompressible fluid past an infinite vertical porous plate in the presence of transverse periodic suction velocity. The problem of combined heat and mass transfer of an electrically conducting fluid in MHD natural convection adjacent to a vertical surface is analyzed by Chen [13] by taking into account the effects of Ohmic heating and viscous dissipation but neglected chemical reaction of the species.

Chaudhury *et al*. [14] have analyzed the effect of radiation on heat transfer in MHD mixed convection flow with simultaneous thermal and mass diffusion from an infinite vertical plate with viscous dissipation and Ohmic heating. The classical model introduced by Cogley *et al*. [14] is used for the radiation effect as it has the merit of simplicity and enables us to introduce linear term in temperature in the analysis for optically thin media. The thermal radiation and Darcian drag force MHD unsteady thermal-convection flow past a semi-infinite vertical plate immersed in a semi-infinite saturated porous regime with variable surface temperature in the presence of transversal uniform magnetic field have been discussed by Ahmed *el al*. [15].

In this paper, it is proposed to study the effects of viscous dissipation and Ohmic dissipation on steady two dimensional magnetohydrodynamic mixed convection heat and mass transfer flow of a Newtonian, electrically conducting and viscous incompressible radiative fluid over a porous vertical embedded in a porous medium plate taking into the account of combined effects of buoyancy force and first-order chemical reaction. The governing equations for this investigation are formulated and solved using perturbation technique.

The governing equations of the Newtonian flow model of electrically conducting radiative and chemically reacting fluid through porous medium in presence of magnetic field with generation and viscous dissipative heat are

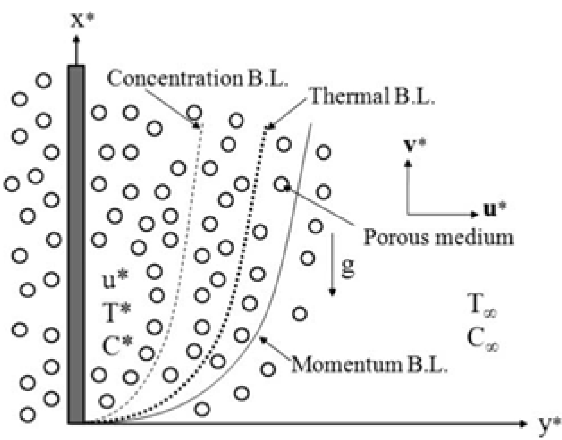

Fig. 5.1 Flow model of the problem

$$\frac{dv^*}{dy^*} = 0 \Rightarrow v^* = -v_0 \text{ (constant)} \tag{1}$$

$$\frac{dp^*}{dy^*} = 0 \Rightarrow p^* \text{ is independent of } y^* \tag{2}$$

$$\rho v^* \frac{du^*}{dy^*} = \mu \frac{d^2u^*}{dy^{*2}} - \frac{\mu}{K^*}u^* - \sigma B_0^2 u^* + \rho g \beta_T \left(T^* - T_\infty\right)$$

$$+ \rho g \beta_C \left(C^* - C_\infty\right) \tag{3}$$

$$\rho C_P v^* \frac{dT^*}{dy^*} = \alpha \frac{d^2u^*}{dy^{*2}} + \mu \left(\frac{du^*}{dy^*}\right)^2 - \frac{dq^*}{dy^*} + \sigma B_0^2 u^{*2} - Q_0\left(T^* - T_\infty\right) \tag{4}$$

$$v^* \frac{dC^*}{dy^*} = D \frac{d^2C^*}{dy^{*2}} - R\left(C^* - C_\infty\right) \tag{5}$$

where, u^* and v^* are the components of dimensional velocities along x^* and y^* directions, respectively, α is the fluid thermal diffusivity. The fourth and fifth terms on RHS of the momentum equation (3) denote the thermal and concentration buoyancy effects, respectively. Also second and fourth terms on the RHS of energy equation (4) represent the viscous dissipation and Ohmic dissipation, respectively. The third and fifth term on the RHS of Eq. (4) denote the inclusion of the effect of thermal radiation and heat absorption effects, respectively.

For the radiative heat flux using the Cogley model [14] is given

$$\frac{\partial q^*}{\partial y^*} = 4\left(T^* - T_\infty\right)I^* \tag{6}$$

where $I^* = \int_0^\infty K_{\lambda w} \frac{\partial e_{b\lambda}}{\partial T^*} d\lambda$

$K_{\lambda w}$ is the absorption coefficient at the wall and $e_{b\lambda}$ is Planck's function.

The appropriate boundary conditions for velocity, temperature and concentration fields are

$$y^* = 0 : u^* = 0, T^* = T_w, C^* = C_w \tag{7}$$

$$y^* \to 0 : u^* \to 0, T^* \to T_\infty, C^* \to C_\infty \tag{8}$$

Where C_w and T_w are the wall dimensional concentration and temperature respectively.

Introducing the following non-dimensional quantities:

$$y = \frac{v_0 y^*}{\nu},\ u = \frac{u^*}{v_0},\ M^2 = \frac{\sigma B_0^2 \nu^2}{\mu v_0},\ K = \frac{K^* v_0^2}{\nu^2},\ \theta = \frac{T^* - T_\infty}{T_w - T_\infty},$$

$$\phi = \frac{C^* - C_\infty}{C_w - C_\infty} \tag{9}$$

Using (6) and (9) in Eqs. (3)–(5), we get the following non-dimensional equations:

$$\frac{d^2u}{dy^2} + \frac{du}{dy} - \left(M + K^{-1}\right)u = -Gr\theta - Gm\phi \tag{10}$$

$$\frac{d^2\theta}{dy^2} + Pr\frac{d\theta}{dy} + PrE\left(\frac{d\theta}{dy}\right)^2 - Pr\left(F + \psi\right)\theta + Pr\,E\,M^2u^2 = 0 \tag{11}$$

$$\frac{d^2\phi}{dy^2} + Sc\frac{d\phi}{dy} - Sc\gamma\phi = 0 \tag{12}$$

where *Gr* is the Grashof number, *Gm* is the solutal Grashoff number, *Pr* is the Prandtl number, *M* is the magnetic field parameter, *F* is the radiation parameter, *Sc* is the Schmidt number, *E* is the Eckert number, ψ is the heat source parameter, and γ is the chemical reaction parameter and *K* is the permeability parameter defined as follows:

$$E=\frac{v_0^2}{C_p\left(T_w-T_\infty\right)},\ Pr=\frac{\mu C_p}{\alpha},\ Gr=\frac{\nu g\beta_T\left(T_w-T_\infty\right)}{\mu v_0^3},$$

$$Gm=\frac{\nu g\beta_C\left(C_w-C_\infty\right)}{\mu v_0^3},\ Sc=\frac{\nu}{D},\ \gamma=\frac{R\nu}{v_0^2},\ \psi=\frac{Q_0\nu}{\rho C_p v_0^2},$$

$$F=\frac{4\nu I^*}{\rho C_p v_0^2}$$

The corresponding boundary conditions in dimensionless form are

$$y=0: u=0, \theta=1, \phi=1$$

$$y\to\infty: u\to 0, \theta\to 0, \phi\to 0$$

Method of Solution

Eqs. (10)–(12) represent a set of partial differential equations that cannot be solved in closed-form. However, these equations can be solved analytically after reducing them to a set of ordinary differential equations in dimensionless form. Thus we can represent the velocity u, temperature θ and concentration ϕ in terms of power of Eckert number E as in the flow of an incompressible fluid Eckert number is always less than unity since the flow due to the Joules dissipation is super imposed on the main flow. Hence, we can assume

$$\begin{aligned} u(y)&=u_0(y)+Eu_1(y)+O\left(E^2\right)\\ \theta(y)&=\theta_0(y)+E\theta_1(y)+O\left(E^2\right)\\ \phi(y)&=\phi_0(y)+E\phi_1(y)+O\left(E^2\right)\end{aligned}\tag{13}$$

Substituting (13) in Eqs. (10)–(12) and equating the coefficient of zeroth powers of Ec (*i.e.* $O(E^0)$), we get the following set of equations:

$$u_0''+u_0'-Nu_0=-Gr\,\theta_0-Gm\ \phi_0\tag{14}$$

$$\theta_0''+Pr\theta_0'-Pr\left(F+\psi\right)\theta_0=0\tag{15}$$

$$\phi_0''+Sc\,\phi_0'-Sc\,\gamma\,\phi_0=0\tag{16}$$

Next, equating the coefficients of first-order of Ec (*i.e.* $O(Ec^1)$), we obtain

$$u_1'' + u_1' - N u_1 = -Gr\,\theta_1 - Gm\,\phi_1 \tag{17}$$

$$\theta_1'' + Pr\theta_1' - Pr\left(F + \psi\right)\theta_1 + Pr\,(u_0')^2 + PrM^2 u_0^2 = 0 \tag{18}$$

$$\phi_1'' + Sc\,\phi_1' - Sc\,\gamma\,\phi_1 = 0 \tag{19}$$

where $N = M^2 + K^{-1}$ and the corresponding boundary conditions are

$$y = 0 : u_0 = 0, u_1 = 0, \theta_0 = 1, \theta_1 = 0, \phi_0 = 1, \phi_1 = 0 \tag{20}$$

$$y \to \infty : u_0 \to 0, u_1 \to 0, \theta_0 \to 0, \theta_1 \to 0, \phi_0 = 0, \phi_1 \to 0 \tag{21}$$

We have restricted the solution of velocity, temperature and concentration fields up to $O(E)$ and neglected the higher order of $O(E^2)$ as the value of $E << 1$. Without going into the details, solutions of Eqs. (17)–(19) with the help of boundary conditions (20) and (21) are obtained as follows:

$$u_0 = A_5\left(e^{-A_4 y} - e^{-A_1 y}\right) + A_6\left(e^{-A_4 y} - e^{-m_1 y}\right) \tag{22}$$

$$\theta_0 = e^{-A_1 y} \tag{23}$$

$$\phi_0 = e^{-m_1 y} \tag{24}$$

$$u_1 = A_{17} e^{-A_4 y} - B_{10} e^{-A_1 y} + B_{11} e^{-2A_1 y} + B_{12} e^{-2A_4 y} - B_{13} e^{-A_{10} y}$$

$$+ B_{14} e^{-2m_1 y} - B_{15} e^{-B_1 y} + B_{16} e^{-B_2 y} \tag{25}$$

$$\theta_1 = B_9 e^{-A_1 y} - B_3 e^{-2A_1 y} - B_4 e^{-2A_4 y} + B_5 e^{-A_{10} y} - B_6 e^{-2m_1 y}$$

$$+ B_7 e^{-B_1 y} - B_8 e^{-B_2 y} \tag{26}$$

$$\phi_1 = 0 \tag{27}$$

The physical quantities of interest are the wall shear stress τ_w is given by

$$\tau_w = \mu \left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = \rho\, v_0^2 u'(0) \tag{28}$$

$$C_{f_x} = \frac{\tau_w}{\rho v_0^2} = u'(0) \tag{29}$$

Using (22), (25) and (28) in (29), we get

$$C_{f_x} = A_6(m_1 - A_4) + A_5(A_1 - A_4) - E\begin{pmatrix} B_{17}A_4 - B_{10}A_1 + 2B_{11}A_1 + 2B_{12}A_4 \\ - B_{13}A_{10} + 2B_{14}m_1 - B_{15}B_1 + B_{16}B_2 \end{pmatrix} \tag{30}$$

The local surface heat flux is given by

$$q_w = -\kappa \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = \rho\, v_0^2 \theta'(0) \tag{31}$$

$$Nu = \frac{q_w}{\rho\, v_0^2} = \theta'(0) \tag{32}$$

Using (23), (26) and (31) in (32), we get

$$Nu = -A_1(1 + EB_9) + E\begin{pmatrix} 2B_3A_1 + 2B_4A_4 - B_5A_{10} \\ + 2B_6m_1 - B_1B_7 + B_2B_8 \end{pmatrix} \tag{33}$$

Validation of the analysis has been performed by comparing the present results with those available in the open literature [13] and a very good agreement has been established, when $K = \infty$, $y = 0.0$, g = 0.0.

In order to verify the accuracy of the present results, we have considered the analytical solutions obtained by Chaudhary *et al*. [13] and computed these solutions for various physical parameters for skin-friction coefficient and local Nusselt number.

Table 5.1 Comparison of present results with those of Chaudhary *et al.* [13] with different values of *F* for C_{fx} ; with $Pr = 0.71$, $Sc = 0.78$, $M = 5.0$, $Gr = 5.0$, $Gm = 5.0$, $E = 0.05$.

Chaudhary *et al.* [13]			Present results		
F	C_{fx}	Nu_x/Re_x	*F*	C_{fx}	Nu_x/Re_x
1.0	1.82701	1.52710	1.0	1.82981	1.52803
2.0	1.77315	1.82047	2.0	1.77918	1.82209
3.0	1.73182	2.17602	3.0	1.73423	2.17803
4.0	1.60718	2.32635	4.0	1.60817	2.32725
5.0	1.52541	2.50981	5.0	1.52598	2.51083

Results and Discussions

To get a physical insight into the problem the numerical evaluation of the analytical results reported in the previous section was performed and a set of results is reported graphically in Figures 5.2 to 5.7 for the cases cooling $Gr>0$ of the plate i.e. free convection currents convey heat away from the plate into the boundary layer. During the numerical calculations the physical parameters are considered as $Pr=0.71$ (diffusing air), $Gr=5$(thermal buoyancy forces are dominant over the viscous hydrodynamic forces in the boundary layer), $F=5>1$(thermal radiation is dominant over the thermal conduction), $E=0.05< 1$(Enthalpy difference is dominant over the kinetic energy).

Figures 5.2 and 5.3 illustrate the influence of the heat absorption and porosity parameters ψ and K, respectively on the flow velocity. The effect is observed on velocity profile by increasing the value of the heat absorption parameter ψ, and the boundary layer thickness decreases with increase in the absorption parameter as shown in Fig. 5.2, which is expected. The opposite trend is observed in Fig. 5.3 for the case when the value of the porous permeability is increased. As depicted in this figure, the effect of increasing the value of porous permeability is to increase the value of the velocity component in the boundary layer due to the fact that drag is reduced by increasing the value of the porous permeability on the fluid flow which results in increased velocity. Fig. 5.4 depicts the effect of radiation on the flow velocity. We note from this figure that there is decrease in the value of flow velocity with increase in radiation parameter F which shows the fact that increase in radiation parameter decrease the velocity in the boundary layer due to decrease in the boundary layer thickness. The effect of chemical reaction parameter γ is highlighted in Fig. 5.5 which shows that the velocity decreases with increasing the rate of chemical reaction γ. Hence increase in the chemical reaction rate parameter leads to a fall in the momentum boundary layer. The trend of the velocity profile in this figure is same as shown in Fig. 5.4. The effect of absorption parameter (ψ) on fluid temperature (θ) is presented in Fig. 5.6. This is due to the fact that the thermal boundary layer absorbs energy which causes the temperature fall considerably with increasing the value of internal heat absorption parameter. The effect the reaction rate parameter (γ) on the species concentration profiles (ϕ) for generative chemical reaction is shown in Fig. 5.7. It is noticed from the graphs that there is a decreasing effect on concentration distribution with increasing the value of the chemical reaction rate parameter in the boundary layer.

Table 1 shows that the skin friction coefficient decreases and local Nusselt number increases with the increasing value of radiation parameter F.

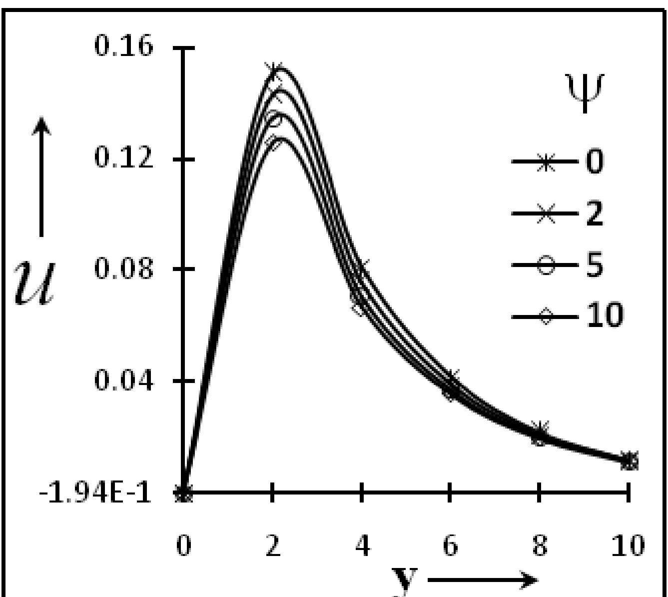

Fig. 5.2 Velocity distribution for heat absorption (ψ)

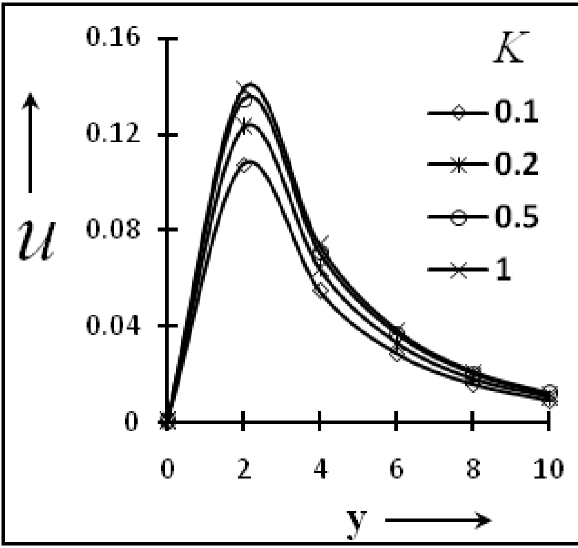

Fig. 5.3 Velocity distribution for porosity (*K*)

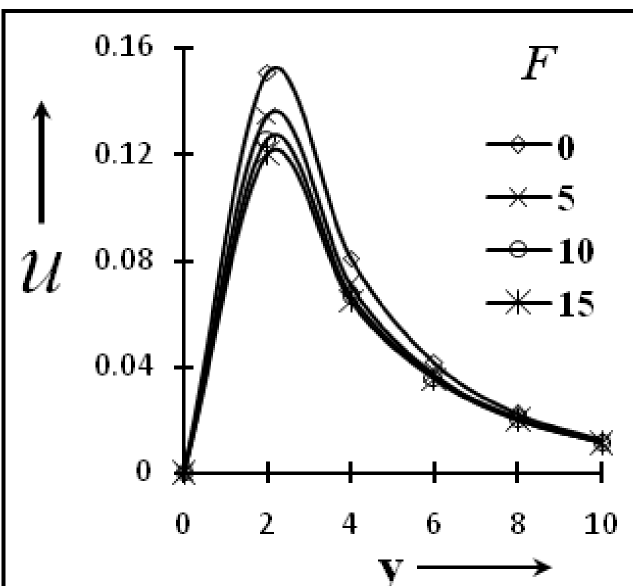

Fig. 5.4 Velocity distribution for radiation (*F*)

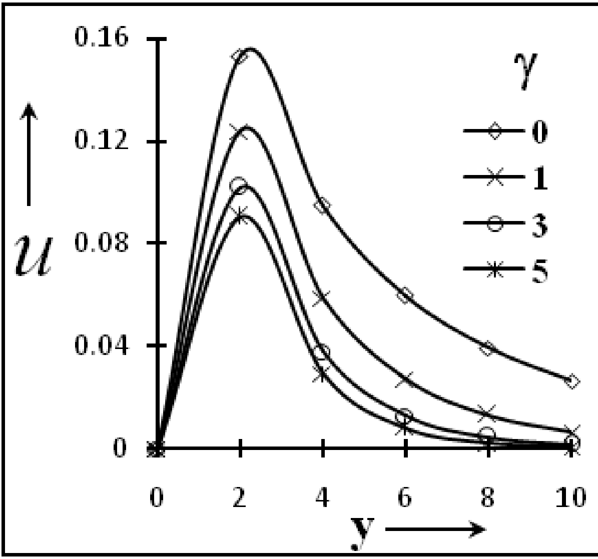

Fig. 5.5 Velocity distribution for chemical reaction (γ)

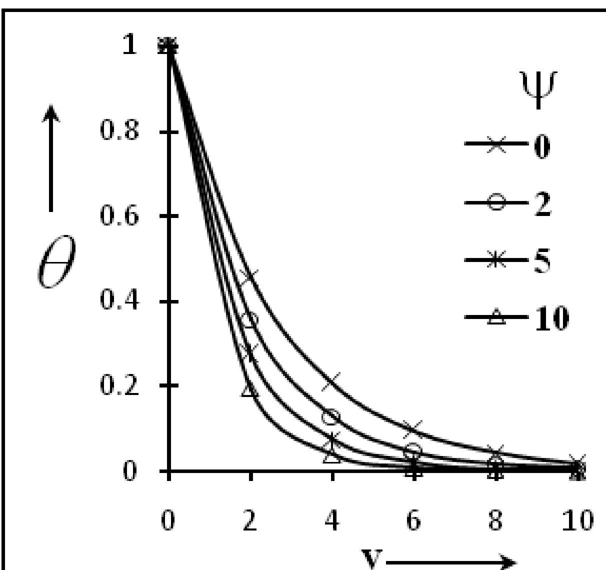

Fig. 5.6 Temperature for heat absorption (ψ)

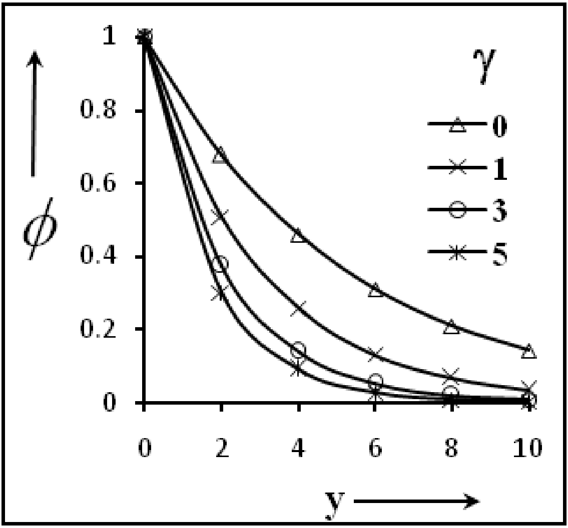

Fig. 5.7 Temperature for chemical reaction (γ)

Conclusions

A theoretical analysis of the steady magnetohydrodynamic flow and mixed convection heat and mass transfer in a viscous, incompressible, electrically-conducting fluid along a semi-infinite vertical plate immersed in a porous medium with thermal radiation has been conducted. The flow model has been setup for homogeneous chemical reaction of first-order in the presence of Ohmic heating and viscous dissipation. The nonlinear and coupled governing equations are solved analytically by perturbation technique. Analytical solutions using the method of complex variables have been derived. The computations have shown that:

i) It is seen that the velocity starts from minimum value of zero at the surface and increases till it attains the peak value and then starts decreasing until it reaches the minimum value at the end of the boundary layer.

ii) Increasing *heat absorption* acts to decelerate the flow velocity in the boundary layer

iii) Flow velocity is accelerated with increasing porosity parameter in the porous regime.

iv) It is seen that with an increase in *heat absorption* of the steady motion, the temperatures are decreased

v) For the steady state case, there is a strong reduction in the concentration distribution for the effect of *generative chemical reaction*.

References:

1. Soundalgekar, V. M., Gupta, S. K., and Birajdar, N. S., Effects of mass transfer and free convection currents on MHD Stokes problem for a vertical plate, *Nuclear Eng. Des*., **53**, 339-346 (1979).
2. Soundalgekar, V. M., Patil, M. R., and Jahagirdar, M. D., MHD Stokes problem for a vertical plate with variable temperature, *Nuclear E ng. Des*., **64**, 39-42 (1981).
3. Khair, K. R., and Bejan A., Mass transfer to natural convection boundary layer flow driven by heat transfer, *Int J Heat Mass Transfer*, **107**, 369–76 (1985).
4. Lin, H. T., and Wu, C. M., Combined heat and mass transfer by laminar natural convection from a vertical plate, *Heat Mass Transfer*, **30**, 369–76 (1995).
5. Yin, K. A., Free convection effect on MHD coupled heat and mass transfer of a moving permeable vertical surface, *Int Commun Heat Mass Transfer*, **26**, 95–104 (1999).
6. Acharya, M., Singh, L. P., and Dash, G. C., Heat and mass transfer over an accelerating surface with heat source in the presence of suction and blowing, *Int J Eng Sci.*, **37**, 189–210 (1999).
7. Muthucumaraswamy, R., and Janakiraman., MHD and Radiation effects on moving isothermal vertical plate with variable mass diffusion, *Theoret. Appl. Mech.*, **33**(1), 17-29 (2006).
8. Hossain, M. A., Alim, M. A., and Rees, D. A. S., The effects of radiation on free convection from a porous vertical plate, *Int. J. Heat and Mass transfer*, **42**(1), 181-191 (1999).
9. Orhan, A., and Ahmet, K., Radiation effect on MHD mixed convection flow about a permeable vertical plate, *Heat and Mass Transfer*, **45**, 239-246 (2008).
10. Ahmed, S. and Liu, I-Chung., Mixed convective three-dimensional heat and mass transfer flow with transversely periodic suction velocity, *Int. J. Applied Mathematics and Mechanics*, **6**, 58-73 (2010).
11. Chen, C. H., Combined heat and mass transfer in MHD free convection from a vertical surface with Ohmic heating and viscous dissipation, *Int J Eng Sci*., **42**, 699–713 (2004).

13. Chaudhary, R. C., Sharma, B. K., and Jha, A. K., Radiation effect with simultaneous thermal and mass diffusion in MHD mixed convection flow, *Rom J Phys.*, **51**(7–8), 715–27 (2006).

14. Cogley, A. C., Vincenty, W. G., and Gilles, E. S., Differential approximation for radiation transfer in a non-gray gas near equilibrium, *AIAAJ*, **6**, 551–5 (1968).

15. Ahmed, S., Batin, A. and Chamkha, A. J., Finite Difference Approach in porous media transport modeling for Magnetohydrodynamic unsteady flow over a vertical plate: Darcian Model, *Int. J. of Numerical Methods for Heat and Fluid Flow*, **24** (4), 1204-1223 (2014).

6

Some common fixed point theorem in metric space for rational expression using integral type mapping

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Abstract

In this paper we prove some fixed point theorems for rational expression satisfying a contractive conditions of integral type in complete metric space. Our results are version of some known results.

Key Words: *Fixed point, Common fixed point, complete metric space, continuous mapping, compatible mapping, rational expression.*

AMS. Classification: 47H10, 54H25

Introduction

Metric fixed point theory is an important mathematical discipline because of its applications in areas as variational and linear inequalities, optimization theory. In 1976 Jungck [2, 3 & 4] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem, which states that "let (X, d) be complete metric space. If T satisfies $d(Tx, Ty) \leq cd\,(x, y)$ for each $x, y\, \epsilon k$ where $0 \leq c < 1$, then has a unique fixed point in, Such that for each

$$x \epsilon X \ , \ lim_{n\to\infty} T^n x = a \quad (1.1)$$

After the classical result, Kannan [5] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theorem dealing with mappings satisfying various type of contractive conditions.

In 2002, Branciari, A[1] and many other mathematician [13-17] analyzed the existence to fixed point for mapping T define on a complete metric space (X, d) satisfying a general contractive condition of integral type.

Preliminaries

Theorem 2.1 (Branciari) : Let (X, d) be a complete metric space c $\in (0, 1)$ and let

$T: X \to X$ be a mapping such that for each $x, y \epsilon X$

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt$$

Where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lesbesgue integrable mapping which is

on, each compact subset of $[0, +\infty)$, non negative & such that for each $\varepsilon > 0, \int_0^{\varepsilon} \varphi(t)dt$, then T has a unique fixed point $a \epsilon X$ such that for each $x \epsilon X, \lim_{n\to\infty} T^n x = a$.

After the paper of Branciari, a fine work has been done by Rhoades [8, 9, 10] extending the result of Brianciari by replacing the condition 2.1.a by the following:

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \int_0^{\max\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\}} \varphi(t)dt \quad (2.1.b)$$

Main Results

The Purpose of this paper is to prove common fixed point theorems by using rational contraction, Rhoades fixed point theorem [9], Branciari result [1]. Juggi's result [17] to compatible maps.

Theorem 3.1: Let T and S be compatible self maps of a complete metric space (X, d) satisfying the following conditions.

$T(x) \subset S(x)$, S is continuous,

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \alpha \int_0^{\max\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\}} \varphi(t)dt$$

$$+\beta \int_0^{\frac{d(Sx,Tx)d(Sy,Ty)}{d(Sx,Sy)}} \varphi(t)dt + \gamma \int_0^{\frac{d(Sx,Ty)d(Sy,Tx)}{d(Sx,Sy)}} \varphi(t)dt$$

$$+\delta \int_0^{\frac{d(Sx,Tx)d(Sx,Ty)+d(Sy,Ty)d(Sy,Tx)}{d(Sx,Sy)}} \varphi(t)dt + \mu \int_0^{d(Sx,sy)} \varphi(t)dt$$

For each $x, y \in X$ with non negative real, $\alpha, \beta, \gamma, \delta, \mu$ such that Where $\psi : R^+ \rightarrow R^+$ is a lesbesgue - integrable mapping which is summable on each compact subset of, non- negative and such that for each

$$\varepsilon > 0, \quad \int_0^{\varepsilon} \varphi(t)dt$$

Then $T\ and\ S$ have a unique common fixed point in X.

Proof:

Let $x_0 \in X$ Since $T(x) \subset S(x)$, Choose $x_1 \in X$ such that $Sx_1 = Tx_0$. In general we construct a sequence x_{n+1} of element of X such that $y_n = Sx_{n+1} = Tx_n\ for$

$n = 0,1,2,3,\dots\dots\dots$ & $y_{n-1} = Sx_n = Tx_{n-1}$ for each integer $n \geq 1$ from 3.1.a

$$\int_0^{d(y_n,y_{n+1})} \Psi(t)dt = \int_0^{d(Tx_n,Tx_{n+1})} \varphi(t)dt$$

$$\therefore \int_0^{d(Tx_n,Tx_{n+1})} \varphi(t)dt \le$$

$$\alpha \int_0^{\max\ \{d(Sx_n,Sx_{n+1}),d(Sx_n,Tx_n),d(Sx_{n+1},Tx_{n+1}),\frac{d(Sx_n,Tx_{n+1})+d(Sx_{n+1}),(Tx_n)}{2}} \varphi(t)dt$$

$$+\beta \int_0^{\frac{d(Sx_n,Tx_n)+d(Sx_{n+1},Tx_{n+1})}{d(Sx_n,Sx_{n+1})}} \varphi(t)dt$$

$$+\delta \int_0^{\frac{d(Sx_n,Tx_n)\, d(Sx_n\,,Tx_{n+1})+d(Sx_{n+1},Tx_{n+1})d(Sx_{n+1},Tx_n)}{d(Sx_n,Sx_{n+1})}} \varphi\,(t)dt$$

$$+\gamma \int_0^{\frac{d(Sx_n,Tx_{n+1})\, d(Sx_{n+1},Tx_n)}{d(Sx_n,Sx_{n+1})}} \varphi\,(t)dt + \mu \int_0^{d(Sx_n,Sx_{n+1})} \varphi\,(t)dt$$

$$\int_0^{d(y_n\,,y_{n+1})} \varphi\,(t)dt \leq$$

$$\alpha \int_0^{\max\{d(y_{n-1},y_n\,),d(y_{n-1},y_n\,),d(y_n,y_{n+1}),\frac{d(y_{n-1},y_{n+1})+d(y_n,y_n)}{2}} \varphi\,(t)dt$$

$$+\beta \int_0^{\frac{d(y_{n-1},y_n\,)\, d(y_n,y_{n+1})}{d(y_{n-1},y_n\,)}} \varphi\,(t)dt \;+\gamma \int_0^{\frac{d(y_{n-1},y_n\,)\, d(y_n,y_n)}{d(y_{n-1},y_n\,)}} \varphi\,(t)dt$$

$$+\delta \int_0^{\frac{d(y_{n-1},y_n\,)d(y_{n-1},y_{n+1})+d\,(y_n,y_{n+1})\, d(y_n,\;y_n)}{d(y_{n-1},y_n\,)}} \varphi\,(t)dt$$

$$+\mu \int_0^{d(y_{n-1},y_n\,)} \varphi\,(t)dt \tag{3.1.c}$$

$$\int_0^{d(y_n\,,y_{n+1})} \varphi\,(t)dt \leq$$

$$\int_0^{\max\{d(y_{n-1},y_n\,),d(y_{n-1},y_n\,),d(y_n,y_{n+1})\,,\frac{d(y_{n-1},y_{n+1}\,)}{2}} \varphi\,(t)dt$$

$$+\beta \int_0^{\frac{d(y_{n-1},y_n\,)\, d(y_n,y_{n+1})}{d(y_{n-1},y_n\,)}} \varphi\,(t)dt$$

$$+\delta \int_0^{d(y_{n-1},y_n\,)d\,(y_n,y_{n+1})} \varphi\,(t)dt \;\; +\mu \int_0^{d(y_{n-1},y_n\,)} \varphi\,(t)dt$$

$$\int_0^{d\,(y_n,y_{n+1})} \varphi\,(t)dt \leq\; (\alpha+\beta+\delta+\mu) \int_0^{d(y_{n-1},y_n\,)} \varphi\,(t)dt$$

$$+(\alpha+\beta+\delta)\int_0^{d\,(y_n,y_{n+1})}\varphi\,(t)dt$$

$$\int_0^{d\,(y_n,y_{n+1})}\varphi\,(t)dt \le \left(\frac{\alpha+\beta+\delta+\mu}{1-\alpha-\beta-\delta}\right)\int_0^{d(y_{n-1},y_n\,)}\varphi\,(t)dt$$

$$\int_0^{d\,(y_n,y_{n+1})}\varphi\,(t)dt \le \left(\frac{\alpha+\beta+\delta+\mu}{1-\alpha-\beta-\delta}\right)^n\int_0^{d(y_0,y_1\,)}\varphi\,(t)dt$$

Let $K=\frac{\alpha+\beta+\delta+\mu}{1-\alpha-\beta-\delta}<$ as $n\to\infty$ we have $lim_{n\to\infty}\int_0^{d\,(y_n,y_{n+1})}\varphi\,(t)\,dt=0$

(3.1.e)

Now we show that $\{y_n\}$ is a Cauchy sequence suppose that it is not. Then $\exists\ an\ E>0$ and subsequence $\{m(p)\}$ and $\{n(p)\}$ such that $<$ with

$$d\big(y_{m\ (p)},y_{n\ (p)}\big)\ge\varepsilon\ ,\ d\ \big(y_{m\ (p)},y_{n\ \ (p)-1}\big)<\varepsilon \qquad (3.1.f)$$

$$d\big(y_{m\ (p)-1},y_{n\ (p)-1}\big)\ <\ d\ \big(y_{m\ (p)-1},y_{m\ (p)}\big)+\ d\ (y_{m\ (p)},y_{n\ (p)-1})$$

$$\therefore\ d\big(y_{m\ (p)-1},y_{n\ (p)-1}\big)\ <\ d\ \big(y_{m\ (p)-1},y_{m\ (p)}\big)+\ \varepsilon$$

$$lim_{p\to\infty}\int_0^{(y_{m\,(p)-1}\ ,y_{m\,(p)-1}\ )}\varphi\,(t)dt=\int_0^{\varepsilon}\varphi\,(t)dt$$

Using (3.1.c, 3.1.f, and 3.1.h) we get

$$\int_0^{\varepsilon}\varphi\,(t)dt\le\int_0^{y_{m\ \ (p)},y_{n\ \ (p)}}\varphi\,(t)dt$$

$$\le f\int_0^{(y_{m\ \ (p)-1},y_{n\ \ (p)-1})}\varphi\,(t)dt\le f\int_0^{\varepsilon}\varphi\,(t)dt$$

Which is contraction, since $f\in[0,1)$ Therefore $\{y_n\}$ is a Cauchy sequence, here converges to $z\in X$ from (3.1.a)

We get

$$\int_0^{d\,(Tz,Tx_n)}\varphi\,(t)dt=\alpha\int_0^{\max\ \{d(Sz\ ,Sx_n\ ),d(Sz\ ,Tz),d(\ Sx_n\ ,Tx_n),\frac{d\ (\ Sz,Tx_n)+d\ (\ Sx_n,Tz\ )}{2}}\varphi\,(t)dt$$

$$+\beta\int_0^{\frac{d(Sz,Tz)d(Sx_n,Tx_n)}{d(Sz,Sx_n)}}\varphi(t)dt+\gamma\int_0^{\frac{d(Sz,Tx_n)d(Sx_n,T_z)}{d(Sz,Sx_n)}}\varphi(t)dt$$

$$+\delta\int_0^{\frac{d(Sz,Tz)\,d(Sz,Tx_n)+d(Sx_n,Tx_n)d(Sx_n,Tz)}{d(Sz,Sx_n)}}\varphi(t)dt$$

$$+\mu\int_0^{d(Sz,Sx_n)}\varphi(t)dt \qquad (3.1.i)$$

Taking limit as $n\rightarrow\infty$ we get

$$\int_0^{d(Tz,Z)}\varphi(t)dt\leq\alpha\int_0^{\max\ [d(Sz,\ z),d(sz,\ Tz),d(z,\ z),\frac{d(Sz,\ z)+d(Z,Tz)}{2}}\varphi(t)dt$$

$$+\beta\int_0^{\frac{d(sz,\ Tz),d(z,\ z))}{d(Sz,\ z)}}\varphi(t)dt+\gamma\int_0^{\frac{d(sz,\ z)\,d(Z,Tz)}{d(Sz,\ z)}}\varphi(t)dt$$

$$+\delta\int_0^{\frac{d(Sz,Tz)d(Sz,\ z)+d(z,z)\,d(Z,Tz)}{d(Sz,\ z)}}\varphi(t)dt+\mu\int_0^{d(Sz,Z)}\varphi(t)dt$$

Which implies $Tz = Z$ and $Sz = Z$.

Now we show that Z is a common fixed point of T and S, T and S are compatible, therefore,

$$lim_{n\to\infty}d\,(Tsx_n,STx_n)=0$$

which since $lim_{n\to\infty}STx_n = Sz$

implies that

$$lim_{n\to\infty}Tsx_n = Sz$$

Now from 3.1.a

$$\int_0^{(Tsx_n,Sx_n)}\varphi(t)dt\leq$$

$$\alpha\int_0^{max\ d(SSx_n,Sx_n),d(SSx_n,TSx_n),d(Sx_n,Tx_n),\frac{d(SSx_n,Tx_n)+d(sx_n,TSx_n)}{2}}\varphi(t)dt$$

$$\beta \int_0^{\frac{d(SSx_n,TSx_n),d(Sx_n,x_n)}{d(SSx_n,Sx_n)}} \varphi(t)dt + \gamma \int_0^{\frac{d(SSx_n,Tx_n),d(Sx_n,Tsx_n)}{d(SSx_n,Sx_n)}} \varphi(t)dt$$

$$+\delta \int_0^{\frac{d(SSx_n,TSx_n),d(SSx_n,Tx_n)+d(Sx_n,Tx_n)d(Sx_n,TSx_n)}{d(SSx_n,Sx_n)}} \varphi(t)dt$$

$$+\mu \int_0^{d(SSx_n,Sx_n)} \varphi(t)dt \qquad (3.1.j)$$

Taking $n \to \infty$

We obtain z = Sz. Again from (3.1.a) we can show that $z = Tz$ and hence z is common fixed point of T and S in

Uniqueness

Let us ω is another fixed point of T and S in X different from z is $z \neq \omega$ then from (3.1.a) we have

$$\int_0^{d(Tw,Tz)} \varphi(t)dt = \alpha \int_0^{\max\ d(Sw,Sz),d(Sw,Tw),d(Sz,Tz)+\frac{d(Sw,Tz)+d[Sz,Tw)}{2}} \varphi(t)dt$$

$$+\beta \int_0^{\frac{d(Sw,Tw),d(Sz,Tz)}{d(Sw,Tw)}} \varphi(t)dt + \gamma \int_0^{\frac{d(Sw,Tz),d(Sz,Tw)}{d(Sw,Sz)}} \varphi(t)dt$$

$$+\delta \int_0^{\frac{d(Sw,Tw)\,d(Sw,Tz)+d(Sz,Tz)\,d(Sw,Tw)}{d(Sw,Sz)}} \varphi(t)dt$$

$$+\mu \int_0^{d(Sw,Sz)} \varphi(t)dt \leq (\alpha+\mu) \int_0^{d(w,z)} \varphi(t)dt$$

$$\therefore \int_0^{d(w,z)} \varphi(t)dt \leq (\alpha+\mu) \int_0^{d(w,z)} \varphi(t)dt$$

Which is Contradiction. Theorem z is unique common fixed point of T and

Remarks

1. Every contractive condition of integral type automatically includes a corresponding contractive condition not involving integrals, by setting $\varphi(t) = 1$ over R^+

2. On setting $\varphi(t) = 1$ and $\beta = \gamma = \delta = 0$, $S = T$ in 3.1 then the result can be significantly improved Rhoades fixed point theorem [8].

Conclusion

We establish a common fixed point theorem in metric space satisfy integral type mapping which shows that our theorem is generalized version of some known theorems.

References

1. Branciari, A. "A fixed point theorem for mapping satisfying a general contractive conditions of integral type, In J. math, math sci 29"9 (2002), 531-536.
2. Jungck, G., commuting mapping and fixed point , Anel, math mapping, 83 (1976), 261-263.
3. Jungck G., compatible mapping and common fixed points, internal. J. math and math Sci, 9 (1986), 771-779.
4. Jungck G. compatible mapping and common fixed point (2) internet. J. math and mat. Sci, 11 (1988), 285-288.
5. Kannan, R., some results on fixed points, Bull. Calcutt math. Soc. 60 (1969), 71-76.
6. Singh, S.L., Hematulin, A. Pant, R., New incidence and common fixed point theorem, applied general topology vol 10, No. 1, (2009), 121-130.
7. Lim, Z.; X, Kang, S.M.; Cho, S.Y., fixed point theorem for mapping satisfying contractive conditions of integral type and application, fixed point theorem application, 2011, 2011:64.
8. Rhoades, RE, Two fixed point theorem for mapping satisfying a general contractive condition of integral type, Int. J. of mathematics and mat. Science. 63, (2003), 400-2013.
9. Rhoades, B.E., contractive definition, non linear analysis, * (1998) 513-526.
10. Rhoades, B.E., some applications of contractive type mapping, mathematics seminar notes vol 5 M (1977) 137-139.
11. Bhardwaj, R., fixed point theorems for compatible mapping satisfying a contractive condition of integral type, Int. J. of mathematical archive-3(5), (2012), 2157-2162.
12. Jaggi, D.S. "unique fixed point theorem, Int. J.P. appl, 8 (1977), 223-230.
13. Pathak H.K., Verma R.K. "Integral type contractive condition for converse commuting mappings. Int. J. of maths analysis, Vol 3, 2009, 0024, 1183-1190.

14. Razuni, A., Moradi, R., “common fixed point theorems of integral type in modular spaces.” Bul. of the Iranian mathematical society Vol 35 No 2 (2009) PP 11-24.

15. Aydi, H., Chauhan, S., Radenovic, S. fixed point of weakly compatible mappings in 4-metric space satisfying common limit Range property;” Facta University vol 28 No. 2 (2013) 197-210.

16. Xeqing Liu, Yan Lu, Shan mih Kang “Fixed point theorems for mappings satisfying contractive conditions of integral type” fixed point theory and applications 2013, 2011:267.

17. Zeqinq, L; Bi, X.; shing, M.K.” two fixed point theorems of mapping satisfying contractive inequities of integral type”, Int. J. of pure s applied mathematics” vol 90, No. 1 (2014), 85-100.

7

On The Efficiency of Chain Ratio Estimator in Sample Surveys

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Abstract

This paper suggests a chain ratio estimator for finite population mean of the study variate using two auxiliary variate under double sampling procedure, when the information on another additional auxiliary variate is available along with the main auxiliary variate. The expressions for bias and mean square error of the asymptotically optimum estimators in the classes are identified in two different cases. The optimum values of the first phase and second phase sample sizes have been obtained for the fixed cost of survey. To illustrate the results, theoretical and empirical studies have also been carried out to demonstrate the efficiency of the proposed estimator with respect to strategies which utilized the information on two auxiliary variates.

Keywords: *Auxiliary variable; chain ratio estimator; bias; mean square error (MSE); efficiency.*

Introduction

Using information on the auxiliary variable x, we often use classical ratio and product estimators depending upon the condition $\rho_{yx} > C_x / 2C_y$

and $\rho_{yx} < -C_x / 2C_y$ respectively, where C_y, C_x denote the coefficients of variation of the variable y and x and ρ_{yx} are the correlation coefficient between y and x. However, in many situations of practical importance, the population mean $\overline{X}$ is not known before the start of a survey. In such a situation, the usual thing to do is to estimate it by the sample mean $\overline{x}_1$ based on a preliminary sample of size n_1 of which n is a subsample $(n < n_1)$. At the most, we use only knowledge of the population mean of another auxiliary character, which is comparatively less correlated to the main characters. That is, if the population mean $\overline{Z}$ of another auxiliary variate Z, closely related to X but compared to X remotely related to Y is known, it is advisable to estimate $\overline{X}$ by $\overline{X} = \overline{x}_1 \overline{Z}/\overline{z}$, which would provide better estimate of $\overline{X}$ than $\overline{x}_1$ to the terms of order $o\left(n^{-1}\right)$ if $\rho_{xz} C_x/C_z > 1/2$.

Chand (1975) and Sukhatme and Chand (1977) proposed a technique of chaining the available information on auxiliary characteristics with the main characteristics. Kiregyera (1980,1984) also proposed some chain type ratio and regression estimators based on two auxiliary variates. In 2002, Al-Jararha and Ahmed defined two classes of estimators by using prior information on parameter of one of the two auxiliary variables under double sampling scheme.

The Proposed Class of Estimator

Let instead of $\overline{X}$ the population mean $\overline{Z}$ of another auxiliary variable Z, which has a positive correlation with x (i.e. $\rho_{xz} > 0$) be known. We further assume $\rho_{yx} > \rho_{yz} > 0$. Let $\overline{x}_1$ and $\overline{z}_1$ be the sample means of x and z respectively based on a preliminary sample of size n_1 drawn with simple random sampling without replacement strategy in order to get an estimate of $\overline{X}$. Then we suggests an estimator for $\overline{Y}$ as

$$T_\alpha = \overline{y}\left[\frac{\frac{n_1\overline{x}_1}{\overline{z}_1}\overline{Z} - n\overline{x}}{(n_1 - n)\frac{\overline{x}_1}{\overline{z}_1}\overline{Z}}\right]^\alpha \tag{1}$$

Where á is determined so as to minimize the mean square (MSE) of $T_{\acute{a}}$, we write

$$\bar{y} = \bar{Y}(1+e_0), \bar{x} = \bar{X}(1+e_1), \bar{x}_1 = \bar{X}(1+e_2)$$

and $\overline{z_1} = \bar{Z}(1+e_3)$

Expressing $T_{\acute{a}}$ in terms of e's, we have

$$T_\alpha = \bar{Y}(1+e_0)\Big[1-\alpha g\left(e_1-e_2+e_3-e_1e_2+e_1e_3-e_2e_3+e_2^2\right)$$

$$+\alpha(\alpha-1)g^2\left(\frac{e_1-e_2+e_3}{3}\right)^2\Big] \tag{2}$$

where $g = \dfrac{n}{n_1 - n}$ and $(1+g) = \dfrac{n_1}{n_1 - n}$

The properties of proposed estimators are obtained for the following two cases.

Case I: When the second phase sample of size n is a subsample of the first phase

of size n^{-1}, we get the following results

$$E(e_0) = E(e_1) = E(e_2) = E(e_3) = 0$$

$$E(e_0^2) = \left(\frac{1}{n} - \frac{1}{N}\right)C_y^2 = M_{0,0},\ E(e_1^2) = \left(\frac{1}{n} - \frac{1}{N}\right)C_x^2 = M_{1,1},$$

$$E(e_2^2) = \left(\frac{1}{n_1} - \frac{1}{N}\right)C_x^2 = M_{2,2},\ E(e_3^2) = \left(\frac{1}{n_1} - \frac{1}{N}\right)C_z^2 = M_{3,3},$$

$$E(e_0e_1) = \left(\frac{1}{n} - \frac{1}{N}\right)\rho_{yx}C_yC_x = M_{0,1},\ E(e_0e_2) = \left(\frac{1}{n_1} - \frac{1}{N}\right)\rho_{yx}C_yC_x = M_{0,2},$$

$$E(e_0e_3) = \left(\frac{1}{n_1} - \frac{1}{N}\right)\rho_{yz}C_yC_z = M_{0,3},\ E(e_1e_2) = \left(\frac{1}{n_1} - \frac{1}{N}\right)C_x^2 = M_{0,2},$$

$$E(e_1e_3)=\left(\frac{1}{n_1}-\frac{1}{N}\right)\rho_{xz}C_xC_z=M_{1,3},\ E(e_2e_3)=\left(\frac{1}{n_1}-\frac{1}{N}\right)\rho_{xz}C_xC_z=M_{2,3}, \qquad (3)$$

Expanding the right hand side of (2) and neglecting the terms of e's greater than second degree, we get

$$T_\alpha-\bar{Y}=\bar{Y}\Big[e_0-\alpha g\begin{pmatrix}e_1-e_2+e_3-e_1e_2+e_1e_3-e_2e_3\\+e_2^2+e_0e_1-e_0e_2+e_0e_3\end{pmatrix}$$

$$+\frac{1}{2}\alpha(\alpha-1)g^2(e_1-e_2+e_3)^2\Big] \qquad (4)$$

Taking the expectations in (4) and using the results of (3) we get the bias of the estimator $T_á$ to the first order of approximation as

$$B(T_\alpha)_I=\bar{Y}\omega[(\omega-g)P/2-Q] \qquad (5)$$

where $\omega=\alpha g$, $P=M_{1,0}-M_{2,0}+M_{3,0}$ and $Q=M_{0,1}-M_{0,2}+M_{0,3}$

Again from equation (4), we have

$$T_\alpha-Y=Y[e_0-\alpha g(e_1-e_2+e_3)] \qquad (6)$$

Squaring both the sides of equation (6), taking expectations and using the results of (3), we obtain the MSE of the estimator $T_á$ to terms of order n^{-1}as

$$M(T_\alpha)_I=\bar{Y}^2\left[M_{0,0}+\omega^2P-2\omega Q\right],\text{ where } \omega=\alpha g. \qquad (7)$$

The MSE of $T_á$ is minimum when

$$\alpha=Q/gP=\alpha_{Iopt}\ \text{(say)} \qquad (8)$$

Putting the value of á from (8) in (7), we get the resulting optimum MSE as

$$M(T_\alpha)_{Iopt}=\bar{Y}^2\left[M_{0,0}-Q^2/P\right] \qquad (9)$$

Remarks (i): When á=0, the estimator $T_á$ in (1) becomes the usual unbiased estimator $\bar{y}$. The bias and MSE of $\bar{y}$ can be obtained by putting $\alpha = 0$ in (5) and (7) respectively as

$$B(\bar{y})_I = 0, \text{ and } M(\bar{y})_I = \bar{Y}^2 M_{0,0} \tag{10}$$

Remark (ii): When á=1, the estimator $T_{á}$ in (1) reduces to the chain dual to ratio estimator $\bar{y}_{dr}^{(c)}$ in double sampling. The bias and MSE of $\bar{y}_{dc}^{(c)}$ can be obtained by putting a = 1 in (5) and (7) respectively as

$$\mathrm{B}\left(\bar{y}_{dr}^{(c)}\right)_I = -\bar{Y} g Q \text{ and } M\left(\bar{y}_{dr}^{(c)}\right)_I = \bar{Y}^2\left[M_{0,0} + g^2 P - 2gQ\right] \tag{11}$$

where $g = \dfrac{n}{n_1 - n}$.

Remark (iii): For a = -1 the estimator T_α in (1) reduces to the chain dual to product estimator $\bar{y}_{dp}^{(c)}$ in double sampling. The bias and MSE of can be obtained by putting in (5) and (7) respectively as

$$B\left(\bar{y}_{dp}^{(c)}\right)_I = \bar{Y} g\left[gP + Q\right] \text{ and}$$

$$M\left(\bar{y}_{dp}^{(c)}\right)_I = \bar{Y}^2\left[M_{0,0} + g^2 P + 2gQ\right] \tag{12}$$

To the first degree of approximation, the variance of chain ratio estimator $\bar{y}_r^{(c)}$ suggested by Chand (1975) in double sampling is given by

$$M\left(\bar{y}_r^{(c)}\right)_I = \bar{Y}^2\left[M_{0,0} + P - 2Q\right] \tag{13}$$

and the chain regression type estimator $y_{reg}^{(dc)}$ suggested by Kiregyera (1984) is given by

$$M\left(\bar{y}_r^{(c)}\right)_I = \bar{Y}^2\begin{bmatrix} M_{0,0} - M_{0,1} C_{yx} \\ + \left(M_{0,2} C_{yx} - 2M_{1,3} C_{yx} C_{yz} + M_{0,3} C_{yz} \rho^2\right) \end{bmatrix} \tag{14}$$

From (10), (11), (12), (13) and (14), it is observed that the proposed class of estimator T_{α} is better than:

(i) the usual unbiased estimator $\bar{y}$, since

$$M\left(\bar{y}\right)-M\left(T_{\alpha}\right)_{Iopt}=\frac{\bar{Y}^{2}Q^{2}}{P}>0$$

(ii) the usual chain ratio estimator $\bar{y}_{r}^{(c)}$, since

$$M\left(\bar{y}_{r}^{(c)}\right)_{I}-M\left(T_{\alpha}\right)_{Iopt}=\frac{\bar{Y}^{2}}{P}(P-Q)^{2}>0$$

(iii) the chain dual to ratio estimator $\bar{y}_{dr}^{(c)}$, since

$$M\left(\bar{y}_{dr}^{(c)}\right)_{I}-M\left(T_{\alpha}\right)_{Iopt}=\frac{\bar{Y}^{2}}{P}(gP-Q)^{2}$$

(iv) the chain dual to product estimator $\bar{y}_{dp}^{(c)}$, since

$$M\left(\bar{y}_{dp}^{(c)}\right)_{I}-M\left(T_{\alpha}\right)_{Iopt}=\frac{\bar{Y}^{2}}{P}(gP+Q)^{2}>0$$

(v) the chain linear regression estimator $y_{reg}^{(dc)}$, since

$$\frac{Q^{2}}{P}>\left[M_{0,1}C_{yx}-M_{0,2}C_{yx}+2M_{1,3}C_{yx}C_{yz}-M_{0,3}C_{yz}\rho_{xz}^{2}\right]$$

Theorem 2.1 The proposed strategy to the first degree of approximation, under optimality condition (8), is always more efficient than,

$M\left(\bar{y}\right)$, $M\left(\bar{y}_{r}\right)$, $M\left(\bar{y}_{dr}^{(c)}\right)$, $M\left(\bar{y}_{dp}^{(c)}\right)$ and more efficient than $M\left(y_{reg}^{(dc)}\right)$ if

$$\left[M_{0,1}C_{yx}-M_{0,2}C_{yx}+2M_{1,3}C_{yx}C_{yz}-M_{0,3}C_{yz}\rho_{xz}^{2}\right]<\frac{Q^{2}}{P}.$$

Case II: When the second phase sample of size n is drawn independently of the first phase sample of size $n_{1.}$ With this condition, we get the following results

$$E(e_{0})=E(e_{1})=E(e_{2})=E(e_{3})=0$$

$$E(e_{0}^{2})=\left(\frac{1}{n}-\frac{1}{N}\right)C_{y}^{2}=M_{0,0},\ E(e_{1}^{2})=\left(\frac{1}{n}-\frac{1}{N}\right)C_{x}^{2}=M_{1,1},$$

$$E(e_2^2)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_x^2=M_{2,2},\ E(e_3^2)=\left(\frac{1}{n_1}-\frac{1}{N}\right)C_z^2=M_{3,3},$$

$$E(e_0e_1)=\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yx}C_yC_x=M_{0,1},\ E(e_2e_3)=\left(\frac{1}{n_1}-\frac{1}{N}\right)\rho_{xz}C_xC_z=M_{2,3},$$

$$E(e_0e_2)=E(e_0e_3)=E(e_1e_2)=E(e_1e_3)=0 \tag{15}$$

Taking the expectation in (4) and using results of (15), we get the bias of the T$_{á}$ up to terms of order n^{-1} as

$$B(T_\alpha)_{II}=\overline{Y}\,g\alpha\left[(\omega-g)P/2-M_{0,1}\right] \tag{16}$$

Again, squaring both the sides of equation (6), taking expectations and using the results of (15), we obtain the MSE of the estimator $T_{á}$ to terms of order n^{-1} as

$$M(T_\alpha)_{II}=\overline{Y}^2\left[M_{0,0}+\omega^2P-2\omega M_{0,1}\right] \tag{17}$$

The MSE of $T_{á}$ is minimum with respect to *á*, if its optimum value is given as

$$\alpha=M_{0,1}/gP=\alpha_{IIopt}\ \text{(say)} \tag{18}$$

where $P=M_{1,0}-M_{2,0}+M_{3,0}$.

Thus, the resulting optimum MSE of $T_{á}$ is given by

$$M(T_\alpha)_{IIopt}=\overline{Y}^2\left[M_{0,0}-(M_{0,1})/P\right] \tag{19}$$

For simplicity we assume that the population size N is large enough as compared to the sample sizes n and n_1 so that the finite population correction (FPC) terms 1/N and 2/N are ignored. Ignoring the FPC in (19), the MSE of $T_{á}$ is given by

$$M(T_\alpha)_{II}=\overline{Y}^2\left[M'_{0,0}+\omega^2P-2\omega M'_{0,1}\right] \tag{20}$$

where $n^{-1}C_y^2=M'_{0,0},\ n^{-1}C_x^2=M'_{1,1}$

$$n_1^{-1} C_x^2 = M'_{2,2} \, , \; n_1^{-1} C_z^2 = M'_{3,3}$$

$$n_1^{-1} \rho_{yx} C_y C_x = M'_{0,1} \, , \; n_1^{-1} \rho_{xz} C_x C_z = M'_{2,3}$$

Minimization of (17) with respect to α, yields its optimum value as

$$\alpha = M'_{0,1} / gP = \alpha^*_{IIopt} \text{ (say)} \tag{21}$$

where $P' = M'_{1,0} - M'_{2,0} + M'_{3,0}$

Putting the value of á from (21) in (20) , we get the resulting MSE as

$$M\,(T_\alpha)_{IIopt} = \overline{Y}^2 \left[M'_{0,0} - (M'_{0,1})^2 / P \right] \tag{22}$$

Remark (i): When a = 0, the estimator T_α in (1) becomes the usual unbiased estimator $\overline{y}$ .Thus putting a = 0 in (20), we get the MSE ofto the first degree of approximation respectively as

$$M\,(\overline{y})_{II} = \overline{Y}^2 M'_{0,0} \tag{23}$$

Remark (ii): For a = 1, the estimator $T_á$ in (1) reduces to the chain dual to ratio estimator $\overline{y}_{dr}^{(c)}$ in double sampling. The MSE of $\overline{y}_{dc}^{(c)}$ can be obtained by putting a = 1 in (20) respectively as

$$M\left(\overline{y}_{dr}^{(c)}\right)_I = \overline{Y}^2 \left[M'_{0,0} + g^2 P' - 2gM'_{0,1} \right] \tag{24}$$

Remark (iii): For a = -1 the estimator $T_á$ in (1) reduces to the chain dual to product estimator $\overline{y}_{dp}^{(c)}$ in double sampling. Setting a = -1 in (20) the MSE of can be obtained respectively as

$$M\left(\overline{y}_{dr}^{(c)}\right)_{II} = \overline{Y}^2 \left[M'_{0,0} + g^2 P' + 2gM'_{0,1} \right] \tag{25}$$

Ignoring the FPC, the variance of chain ratio estimator $\overline{y}_r^{(c)}$ suggested by Chand (1975) in double sampling is given by

$$M\left(\overline{y}_r^{(c)}\right)_{II} = \overline{Y}^2 \left[M'_{0,0} - 2M'_{0,1} + P' \right] \tag{26}$$

And the chain regression type estimator $y_{reg}^{(dc)}$ suggested by Kiregyera (1984) in double sampling is

$$M\left(\overline{y}_{reg}^{(c)}\right)_{II} = \overline{Y}^2\left[M'_{0,0} - M'_{0,1}C_{yx}\right] \tag{27}$$

From (23), (24), (25), (26) and (27), it is observed that the asymptotically optimum estimator (AOE) $(T_\alpha)_{IIopt}$ is better than:

(i) the usual unbiased estimator $\overline{y}$, if

$$M\left(\overline{y}\right)_{II} - M\left(T_\alpha\right)_{IIopt} = \frac{\overline{Y}^2\left(M'_{0,1}\right)^2}{P'} > 0$$

(ii) the usual chain ratio estimator $\overline{y}_r^{(c)}$, if

$$M\left(\overline{y}_r^{(c)}\right)_{II} - M\left(T_\alpha\right)_{IIopt} = P'\left(M'_{0,1} - 1\right)^2 > 0$$

(iii) the chain dual to ratio estimator $\overline{y}_{dr}^{(c)}$, since

$$M\left(\overline{y}_{dr}^{(c)}\right)_{II} - M\left(T_\alpha\right)_{IIopt} = \frac{\overline{Y}^2}{P'}\left(gP' - M'_{0,1}\right)^2 > 0$$

(iv) the chain dual to product estimator $\overline{y}_{dp}^{(c)}$, since

$$M\left(\overline{y}_{dp}^{(c)}\right)_{II} - M\left(T_\alpha\right)_{IIopt} = \frac{\overline{Y}^2}{P'}\left(gP' + M'_{0,1}\right)^2 > 0$$

(v) the chain linear regression estimator , if

$$\frac{\left(M'_{0,1}\right)^2}{P'} > \left[M'_{0,1}C_{yx}\right]$$

Theorem 2.2: To the terms of order n^{-1}, the proposed estimator under optimality condition (18), is always more efficient than $M\left(\overline{y}\right)$,

$$M\left(\overline{y_r}\right), M\left(\overline{y}_{dr}^{(c)}\right), M\left(\overline{y}_{dp}^{(c)}\right)$$

and more efficient than $M(y_{reg}^{(dc)})$ if

$$\frac{\left(M'_{0,1}\right)^2}{P'} > \left[M'_{0,1}C_{yx}\right]$$

3. The proposed estimator has so far been compared with other estimators with respect to their variances. However in practical applications, the cost aspect should also be taken into account. Therefore, we have to fix the total cost of the survey and and then have to find optimum sizes of preliminary and final samples so that the variance of the estimator is minimized. In this section, we shall consider the cost of the survey and find the optimum sizes of the preliminary and second-phase samples in **Case I** and ***Case II*** separately.

Case I:

When one auxiliary variate x is used then the cost function is given by $C = nC_1 + n_1C_2$, where C is the total cost, C_1 is the cost per unit of collecting information on the study variate y and C_2 is the cost per unit of collecting auxiliary variate x respectively of the survey.

When we use additional auxiliary variate z to estimate T_α, then the cost function is given by

$$C = {}^{n}C_1 + n_1(C_2 + C_3) \tag{28}$$

where C_3 is the cost per unit of collecting information on the auxiliary variate z. Ignoring FPC, the MSE of (T_α) in (7) can be expressed as

$$M(T_\alpha)_I = \frac{1}{n}V_1 + \frac{1}{n_1}V_2$$

whre $V_1 = \overline{Y}^2 (M'_{0,0} - 2\omega M'_{0,1} + \omega^2 M'_{1,0})$

$$V_2 = \overline{Y}^2 \left\{M'_{0,3} + 2\omega M'_{0,2} + \omega^2 (M'_{3,0} - M'_{2,0})\right\}$$

$$\left(\frac{1}{n_1}\right)\rho_{yz}C_yC_z = M'_{0,3}.$$

Since it is assumed that $C_1 > C_2 > C_3$, the optimum values of n and n_1 for fixed cost $C = C_0$ which minimizes the variance of T_α at (7) under cost function are given by

$$n_{opt.} = \frac{C_0 \sqrt{V_1/C_1}}{\sqrt{V_1 C_1} + \sqrt{V_2 (C_2 + C_3)}}$$

$$n_{1opt.} = \frac{C_0 \sqrt{V_2/(C_2 + C_3)}}{\sqrt{V_1 C_1} + \sqrt{V_2 (C_2 + C_3)}}$$

Hence, the resulting MSE of T_α is given by

$$M_{opt}(T_\alpha)_I = \frac{1}{C_0}\left[\sqrt{V_1 C_1} + \sqrt{V_2 (C_2 + C_3)}\right] \tag{29}$$

If all the resources were diverted towards the study variate y only, then the optimum sample size would be

$$n^{**} = C/C_1$$

Thus, the MSE of sample mean $\bar{y}$ for a given fixed cost $C = C_0$ in case of large population is given by

$$M_{opt}(\bar{y}) = \frac{C_1}{C_0} S_y^2 \tag{30}$$

Now, from (29) and (30), the proposed sampling strategy would be profitable if

$$M_{opt.}(T_\alpha)_I < M_{opt.}(\bar{y}) \text{ or } \frac{C_2 + C_3}{C_1} < \left[\frac{S_y - \sqrt{V_1}}{\sqrt{V_2}}\right]^2 .$$

Case II: Suppose y is measured on n units, x and z are measured on n_1. Then, we consider a simple cost function

$$C = nC_1 + n_1(C_2' + C_3') \tag{31}$$

Where C_2' denote the cost per unit of observing x and C_3' denote the cost per unit of observing z values respectively. Thus, the MSE of T_α at the equation (20) can be written as

$$M_{opt}(T_\alpha)_{II} = \frac{1}{n}V_1 + \frac{1}{n_1}V_3 \tag{32}$$

where $V_3 = \overline{Y}^2 \omega^2 (M'_{3,0} - M'_{2,0})$.

Now, to obtain the optimum allocation of sample between phases for a fixed cost $C = C_0$, we minimize the above equation (32) with condition (31). It is easily found that this minimum is attained for

$$n_{opt.} = \frac{C_0\sqrt{V_1/C_1}}{\sqrt{V_1C_1} + \sqrt{V_3(C_2' + C_3')}}, \quad n_{1opt.} = \frac{C_0\sqrt{V_3/(C_2' + C_3')}}{\sqrt{V_1C_1} + \sqrt{V_3(C_2' + C_3')}}.$$

Hence, the optimum MSE corresponding to these optimum values of n and n_1 is given by

$$M_{opt}(T_\alpha)_{II} = \frac{1}{C_0}\left[\sqrt{V_1C_1} + \sqrt{V_2(C_2' + C_3')}\right]^2 \tag{33}$$

From equations (30) and (33) it is observed that the proposed estimator T_α yields less MSE or variance than that of sample mean $\overline{y}$ for the same fixed cost if

$$\frac{C_2' + C_3'}{C_1} < \left[\frac{S_y - \sqrt{V_1}}{\sqrt{V_3}}\right]^2 . \frac{C_2 + C_3}{C_1} < \left[\frac{S_y - \sqrt{V_1}}{\sqrt{V_2}}\right]^2 .$$

Empirical Study

In this section we illustrate the performance of the proposed estimator T_α over various other estimators $\bar{y}$, $\overline{y}_r^{(c)}$, $\overline{y}_{dr}^{(c)}$, $\overline{y}_{dp}^{(c)}$ and $y_{reg}^{(dc)}$ through four populations of natural data given as

Population I (Source: Cochran [1977])

Y: Number of 'placebo' children

X: Number of paralytic polio cases in the placebo group

Z: Number of paralytic polio cases in the 'not inoculated' group

N=34,n=10,n_1=15,

$\overline{Y}$=4.92 acre, $\overline{X}$=2.59 acre, $\overline{Z}$=2.91 acre,

$\rho_{yx} = 0.7326, \rho_{yz} = 0.6430,$

$\rho_{xz} = 0.6837, C_y^2 = 1.0248,$

$C_x^2 = 1.5175, C_z^2 = 1.1492$

Population II (Source: Murthy [1967])

Y: Area under wheat in 1964

X: Area under wheat in 1963

Z: Cultivated area in 1961

$N = 34, n = 7, n_1 = 10, \overline{Y} = 199.44$ acre, $\overline{X} = 208.89$ acre,

$\overline{Z} = 747.59$ acre,

$\rho_{yx} = 0.9801, \rho_{yz} = 0.9043,$

$\rho_{xz} = 0.9097, C_y^2 = 0.5673, C_x^2 = 0.5191$

and $C_z^2 = 0.3527$

Population III (Source: Srivastava *et al.* [1989, Page 3922])

Y: The measurement of weight of children

X: Mid arm circumference of children

Z: Skull circumference of children

$N = 82, n = 25, n_1 = 43, \overline{Y} = 5.60$ kg, $\overline{X} = 11.90$ cm,

$\overline{Z} = 39.80$ cm,

$\rho_{yx} = 0.09, \rho_{yz} = 0.12, \rho_{xz} = 0.86,$

$C_y^2 = 0.0107, C_x^2 = 0.0052$ and $C_z^2 = 0.0008$

Population IV (Source: Srivastava *et al.* [1989, Page 3922])

Y: The measurement of weight of children

X: Mid arm circumference of children

Z: Skull circumference of children

$N = 55, n = 18, n_1 = 30, \bar{Y} = 17.08$ kg, $\bar{X} = 16.92$ cm, $\bar{Z} = 50.44$ cm,

$\rho_{yx} = 0.54, \rho_{yz} = 0.51, \rho_{xz} = -0.08,$

$C_y^2 = 0.0161, C_x^2 = 0.0049$ and

$C_z^2 = 0.0007$

To observe the relative performance of different estimators of $\bar{Y}$, we have computed the percentage relative efficiencies (PRE) of the proposed estimator T_α, the usual chain ratio estimator $\bar{y}_r^{(c)}$, chain dual to ratio estimator $\bar{y}_{dr}^{(c)}$, chain dual to product estimator $\bar{y}_{dp}^{(c)}$ and linear regression estimator $y_{reg}^{(dc)}$ with respect to the usual unbiased estimator $\bar{y}$ and the results are shown in Table 7.1.

Table 7.1 Percentage relative efficiencies of different estimators with respect to $\bar{y}$

Estimator	$\bar{y}$	$\bar{y}_r^c$	$\bar{y}_{dr}^c$	$\bar{y}_{dp}^c$	$\bar{y}_{reg}^{(dc)}$	T_α
Case I						
Population I	100	136.93	32.83	10.76	180.78	188.96
Population II	100	718.47	78.54	11.47	1823.81	748.22

Population IV	100	132.01	127.34	47.43	123.65	132.45
Estimator	$\bar{y}$	$\bar{y}_r^c$	$\bar{y}_{dr}^c$	$\bar{y}_{dp}^c$	$\bar{y}_{reg}^{(dc)}$	T_α
Case II						
Population I	100	218.26	41.69	10.49	295.02	278.03
Population II	100	4731.51	66.48	9.75	2530.28	5195.13
Population III	100	86.75	73.36	58.42	100.82	101.44
Population IV	100	87.13	75.05	75.04	100.01	100.01

Conclusions

The chain ratio estimator in double sampling has been analyzed and its bias and MSE equations have been obtained in two different cases. The MSE of the proposed estimator has beencompared with the MSEs of the usual unbiased estimator $\bar{y}$, chain ratio estimator $\bar{y}_r^{(c)}$, chain dual to ratio estimator $\bar{y}_{dr}^{(c)}$, chain dual to product estimator $\bar{y}_{dp}^{(c)}$ and linear regression estimator $y_{reg}^{(dc)}$ on a theoretical basis and also conditions for obtaining minimum MSE has been derived. The percentage relative efficiencies of different estimators with respect to have been computed and is shown in Table 1. From Table 1, it clearly indicates that the efficiency of the proposed estimator is superior to the estimators $\bar{y}_r^{(c)}$, $\bar{y}_{dr}^{(c)}$, $y_{reg}^{(dc)}$, and in both the cases except for the data set of population II for linear regression estimator in Case I and population I in Case II. Thus, the proposed estimator or $\left[(T_\alpha)_{Iopt} \text{ and } (T_\alpha)_{IIopt}\right]$ is preferred to use in practice.

References

1. Al-Jararha, J., Ahmed, M. S., The class of chain estimator for a finite population variance using double sampling ., *Information and Managment Sciences,* **13**(2), 13-18 (2002).

2. Bose, C., Note on the sampling error in the method of double sampling., *Sankhya,* **6**, 330(1943).

3. Chand, L., Some ratio type estimator based on two or more auxiliary variables.,*Unpublished Ph. D. dissertation, Lowa State University, Ames, Lowa,*(1975).

4. Choudhury, S., Singh, B.K., A class of chain ratio –cum-dual to ratio type estimator with two auxiliary characters under double sampling in sample surveys.,*Statistics in Transition-new series,* **13**(3), 519-536(2012).

5. Kadilar, C., Cingi, H., A study on the chain ratio-type estimator., *Hacettepe Journal of Mathematics and Statis tics* ,**32,** 105-108(2003) [6] Kiregyera, B., Regression type estimators us ing two auxiliary variables and the model of double sampling from finite population., *Metrika,* **31**, 215-226(1984).

7. Kumar, M., BahL, S., Class of dual to ratio estimators for double sampling., *Statistical Papers,* **47**, 319-326(2006).

8. Pradhan, B. K. (2005). A chain regression estimator in two phase sampling using multi-auxiliary information., *Bulletin of the Malaysian Mathematical Sciences Society,* (**2**), 28(1), 81–86.

9. Singh, H.P., Mathur, N., Chandra, P., A chain-type estimator for population variance using auxiliary variables in two-phase sampling., *Statistics in Transition-new series*,**10**(1), 75-84 (2009).

10. Singh, V.K., Singh, B.K., Singh, G.N., An efficient class of dual to ratio estimators using two auxiliary characteristics., *Journal Of Scientific Research* **43**, 219-228(1993)

11. Singh, V.K., Singh, G.N., Shukla, D., A class of chain ratio type estimators with two auxiliary variables under double sampling scheme.,*Sankhya: The Indian Journal of Sta tistics* **56**, Series B 2, 209-221(1994).

12. Srivenkataramana, T., A dual to ratio estimator in sample surveys., *Biometrika,* **67**(1), 199-204(1980).

13. Srivastava, S. K., Jhajj, H. SA., A class of estimators using auxiliary information for estimating finite population variance., *Shankhya,* **42**(1–2), 87–96 (1980).

14. Srivastava, S. R., Khare, B.B., Srivastava, S.R., A generalized chain ratio estimator for mean of finite population.,*Journal of Indian Society Of Agricultural Statistics,* **42**(I), 108-117(1990).

15. Sukhatme, B. V., Chand, L., Multivariate ratio-type estimators., *Proceedings of Social Statitics Section, American Statistical Association,* 927–931(1977).

8

Mathematical Modelling for Mixed Convection MHD Flow over a permeable semi-infinite vertical moving Plate in a Porous Regime with Heat Absorption

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Abstract

In this analysis MHD boundary layer flow with heat and mass transfer towards a semi-infinite vertical permeable moving plate embedded in a Darcian porous medium subject to time-dependent wall suction in presence of heat absorption are presented. A uniform transverse magnetic field to the wall with thermal and concentration buoyancy effects is considered. The plate is assumed to move with a constant velocity in the direction of fluid flow while the free stream velocity is assumed to follow the exponentially increasing small perturbation law. The dimensionless governing equations for this investigation are solved analytically using two-term *harmonic* and *non-harmonic* functions. The obtained analytical results reduce to previously published results on a special case of the problem. Analytical solutions of these equations are obtained by perturbation method. It is found that both the velocity and temperature distributions decrease as heat absorption increases. An increase in concentration buoyancy effects leads to increase in fluid velocity.

***Keywords**: Heat absorption; Thermal buoyancy; Unsteady flow; MHD; Mixed convection flow; Vertical porous plate; Perturbation method.*

Introduction

Magnetohydrodynamic transient convective flow with heat and mass transfer has been a subject of interest of many researchers because of its varied applications in science and technology, which involve the interaction of several phenomena. The phenomena of free convection arise in the fluid when temperature changes cause density variation leading to buoyancy forces acting on the fluid elements. In natural process and industrial applications many transport processes exist where transfer of heat and mass takes place simultaneously as a result of combined buoyancy effects of thermal diffusion and diffusion of chemical species. The process of heat and mass transfer is encountered in fluid fuel nuclear reactor, chemical process industries and many engineering applications in which fluid is considered to be the working medium. Chen and Strobel [1] analyzed the nature of convective flows resulting from buoyancy induced pressure gradient in a laminar boundary layer of a stretched sheet with constant velocity and temperature. Soundalgekar [2] analyzed the viscous dissipation effects on unsteady free convective flows past an infinite vertical porous plate with constant suction. It was assumed that the plate temperature oscillates in such a way that its amplitude is small. An effect of heat and mass transfer on MHD free convection along a moving permeable vertical surface has been studied by Abdelkhalek transverse sinusoidal suction velocity and a constant free stream velocity was presented by Ahmed [9]. Also, Ahmed and Liu [10] analyzed the effects of mixed convection and mass transfer of three-dimensional oscillatory flow of a viscous incompressible fluid past an infinite vertical porous plate in presence of transverse sinusoidal suction velocity oscillating with time and a constant free stream velocity. Ahmed [11] investigated the effect of periodic heat transfer on unsteady MHD mixed convection flow past a vertical porous flat plate with constant suction and heat sink when the free stream velocity oscillates in about a non-zero constant mean. Chamkha [12] considered the problem of steady, hydromagnetic boundary layer flow over an accelerating semi-infinite porous surface in the presence of thermal radiation, buoyancy and heat generation or absorption effects.

In fact, flows of fluids through porous media have possible applications in many branches of science and technology. In fact, flows of fluids through porous media have attracted the attention of a number of scholars because of their possible applications in many branches of science and technology. Classically the Darcian model is used to simulate the bulk effects of porous materials on flow dynamics and is valid for Reynolds numbers based on the pore radius, up to approximately 10. Lai and Kulacki [13] has investigated coupled heat and mass transfer by mixed convection from an isothermal vertical plate in a porous medium. Chamkha [14] studied the transient-free convection magnetohydrodynamic boundary layer flow in a fluid-saturated porous medium channel. B´eg *et. al.* [15] presented perturbation solutions for the transient oscillatory hydromagnetic convection in a Darcian porous media with a heat source present. Ahmed and Kalita [16] presented the magnetohydrodynamic transient convective radiative heat transfer one-dimensional flow in an isotropic, homogenous porous regime adjacent to a hot vertical plate.

The objective of this paper is to consider unsteady simultaneous convective heat and mass transfer flow along a vertical permeable plate embedded in a fluid-saturated porous medium in the presence of mass blowing or suction, magnetic field effects, and absorption effects. Most of previous works assumed that the semi-infinite plate is at rest. In the present work, it is assumed that the plate is embedded in a uniform porous medium and moves with a constant velocity in the flow direction in the presence of a transverse magnetic field. It is also assumed that the free stream to consist of a mean velocity and temperature over which are superimposed an exponentially varying with time.

Problem formulation

Consider unsteady two-dimensional flow of a laminar, incompressible, viscous, electrically conducting and heat-absorbing fluid past a semi-infinite vertical permeable moving plate embedded in a uniform porous medium and subjected to a uniform transverse magnetic field in the presence of thermal and concentration buoyancy effects (see Fig. 1). It is assumed that there is no applied voltage which implies the absence of an electrical field. The transversely applied magnetic field and magnetic Reynolds number are assumed to be very small so that the induced magnetic field and the Hall effect are negligible [20]. A consequence of the small magnetic Reynolds number is the uncoupling of the Navier–Stokes equations from Maxwell's equations [18]. The governing equations for this investigation are based on the balances of mass, linear momentum, energy and concentration species. Taking into consideration the assumptions made above, these equations can be written in Cartesian frame of reference, as follows:

$$\frac{\partial v^*}{\partial y^*} = 0 \tag{1}$$

$$\frac{\partial u^*}{\partial t^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu \frac{\partial^2 u^*}{\partial y^{*2}} + g\beta_T(T^* - T^*_\infty) + g\beta_C(C^* - C^*_\infty) - \frac{\mu}{K^*}u^* - \frac{\sigma B_0^2}{\rho}u^* \tag{2}$$

$$\frac{\partial T^*}{\partial t^*} + v^* \frac{\partial T^*}{\partial y^*} = \alpha \frac{\partial^2 T^*}{\partial y^{*2}} - \frac{Q_0}{\rho C_P}(T^* - T^*_\infty) \tag{3}$$

$$\frac{\partial C^*}{\partial t^*} + v^* \frac{\partial C^*}{\partial y^*} = D \frac{\partial^2 C^*}{\partial y^{*2}} \tag{4}$$

where x^*, y^*, and t^* are the dimensional distances along and perpendicular to the plate and dimensional time, respectively, u^* and v^* are the components of dimensional velocities along x^* and y^* directions, respectively, ρ is the fluid density, μ is the kinematic viscosity, C_P is the specific heat at constant pressure, σ is the fluid electrical conductivity, B_0 is the magnetic induction, K^* is the permeability of the porous medium, T^* is the dimensional temperature, Q_0 is the dimensional heat absorption coefficient, C^* is the dimensional concentration, a is the fluid thermal diffusivity, D is the mass diffusivity, g is the gravitational acceleration, and β_T and β_c are the thermal and concentration expansion coefficients, respectively. The magnetic and viscous dissipations are neglected in this study. The third and fourth terms on the RHS of the momentum Eq. (2) denote the thermal and concentration buoyancy effects, respectively. Also, the last term of the energy Eq. (3) represents the heat absorption effects. It is assumed that the permeable plate moves with a constant velocity in the direction of fluid flow, and the free stream velocity follows the exponentially increasing small perturbation law. In addition, it is assumed that the temperature and the concentration at the wall as well as the suction velocity are exponentially varying with time.

With foregoing assumptions, the appropriate boundary conditions for the velocity, temperature and concentration fields are

$$y^* = 0: \;\; u^* = u^*_p, \;\; T^* = T^*_W + \varepsilon(T^*_W - T^*_\infty)e^{n^*t^*}, \;\; C^* = C^*_W + \varepsilon(C^*_W - C^*_\infty)e^{n^*t^*} \tag{5}$$

$$y^* \to \infty: \;\;\; u^* \to U^*_\infty\left(1 + \varepsilon e^{n^*t^*}\right), \;\; T^* \to T^*_\infty, \;\; C^* \to C^*_\infty \tag{6}$$

where u^*_p, C^*_w and T^*_w are the wall dimensional velocity, concentration and temperature, respectively. U_∞^*, C_∞^* and T_∞^* are the free stream dimensional velocity, concentration and temperature, respectively, U_0 and n^* are constants.

It is clear from Eq. (1) that the suction velocity at the plate surface is a function of time only. Assuming that it takes the following exponential form:

$$v^* = -V_0\left(1 + \varepsilon A e^{n^* t^*}\right) \tag{7}$$

where A is a real positive constant, e and eA are small less than unity, and V_0 is a scale of suction velocity which has non-zero positive constant. Outside the boundary layer, Eq. (2) gives

$$-\frac{1}{\rho}\frac{d\rho^*}{\partial x^*} = \frac{dU_\infty^*}{dt^*} + \frac{\nu}{K^*}U_\infty^* + \frac{\sigma}{\rho}B_0^2 U_\infty^* \tag{8}$$

It is convenient to employ the following dimensionless variables:

$$\eta = \frac{V_0 y^*}{\nu},\ u = \frac{u^*}{U_0},\ v = \frac{v^*}{U_0},\ U_\infty = \frac{U_\infty^*}{U_0},\ U_p = \frac{U_p^*}{U_0},\ t = \frac{V_0^2 t^*}{\nu},\ n = \frac{\nu n^*}{V_0^2},$$

$$M = \frac{\sigma B_0^2 \nu}{\mu V_0^2},\ K = \frac{K^* V_0^2}{\nu^2},\ \theta = \frac{T^* - T_\infty^*}{T_w^* - T_\infty^*},\ C = \frac{C^* - C_\infty^*}{C_w^* - C_\infty^*},\ Pr = \frac{\rho\nu C_P}{\kappa} = \frac{\nu}{\alpha},$$

$$G_T = \frac{\rho g \beta_T (T_w^* - T_\infty^*)}{U_0 V_0^2},\ G_c = \frac{\rho g \beta_C (C_w^* - C_\infty^*)}{U_0 V_0^2},\ Sc = \frac{\nu}{D},\ \phi = \frac{Q_0 \nu}{\rho C_P V_0^2} \tag{9}$$

In view of (7) to (8) and the above non-dimensional variables (9), the eqs. (2) to (4) can be expressed in non-dimensional form as

$$\frac{\partial u}{\partial t} - (1 + A\varepsilon e^{nt})\frac{\partial u}{\partial \eta} = G_T\theta + G_c C + \frac{\partial U_\infty}{\partial t} + \frac{\partial^2 u}{\partial \eta^2} + N(U_\infty - u) \tag{10}$$

$$\frac{\partial \theta}{\partial t} - (1 + A\varepsilon e^{nt})\frac{\partial \theta}{\partial \eta} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} - \phi\theta \tag{11}$$

$$\frac{\partial C}{\partial t} - (1 + A\varepsilon e^{nt})\frac{\partial C}{\partial \eta} = \frac{1}{Sc}\frac{\partial^2 C}{\partial \eta^2} \tag{12}$$

where $N = M + K^{-1}$ and G_c, G_T, Pr, ϕ and Sc are the solutal Grashof number, thermal Grashof number, Prandtl number, dimensionless heat absorption coefficient, and the Schmidt number, respectively. By setting G_c and ϕ equal to zero and ignoring Eq. (12), Eqs. (10) and (11) reduce to those reported by Kim [19].

Non-dimensional boundary conditions are

$$u = U_p, \ \theta = 1 + \varepsilon e^{nt}, \ C = 1 + \varepsilon e^{nt} \ \text{at} \ \eta = 0 \tag{13}$$

$$u \to U_\infty, \ \theta \to 0, \ C \to 0 \quad \text{as} \ \eta \to \infty \tag{14}$$

Methodology

Eqs. (10)–(12) represent a set of partial differential equations that can not be solved in closed form. However, it can be reduced to a set of ordinary differential equations in dimensionless form that can be solved analytically. This can be done by representing the velocity, temperature and concentration as

$$u(\eta) = f_0(\eta) + \varepsilon e^{nt} f_1(\eta) + O(\varepsilon^2) \tag{15}$$

$$\theta(y) = g_0(y) + \varepsilon e^{nt} g_1(y) + O(\varepsilon^2) \tag{16}$$

$$C(y) = h_0(y) + \varepsilon e^{nt} h_1(y) + O(\varepsilon^2) \tag{17}$$

Substituting Eqs. (15)–(17) into Eqs. (10)–(12), equating the harmonic and non-harmonic terms, and neglecting the higher-order terms of $O(\varepsilon^2)$, one obtains the following pairs of equations for (f_0, g_0, h_0) and (f_1, g_1, h_1).

$$f_0'' + f_0' - N f_0 = -N - G_T g_0 - G_c h_0 \tag{18}$$

$$f_1'' + f_1' - (N + n) f_1 = -(N + n) - A f_0' - G_T g_1 - G_c h_1 \tag{19}$$

$$g_0'' + Pr g_0' - Pr \phi g_0 = 0 \tag{20}$$

$$g_1'' + Pr g_1' - n Pr g_1 - Pr \phi g_1 = -A Pr g_0' \tag{21}$$

$$h_0'' + Sc h_0' = 0 \tag{22}$$

$$h_1'' + Sc h_1' - n Sc h_1 = -A Sc h_0' \tag{23}$$

where a prime denotes ordinary differentiation with respect to η. The corresponding boundary conditions can be written as

$$\eta = 0: \quad f_0 = U_P, \ f_1 = 0, \ g_0 = 1, \ g_0 = 1, \ h_0 = 1, \ h_0 = 1 \tag{24}$$

$$\eta \to \infty: \quad f_0 \to 1, \ f_1 \to 1, \ g_0 \to 0, \ g_1 \to 0, \ h_0 \to 0, \ h_1 \to 0 \tag{25}$$

The solutions of Eqs. (18)–(23) subject to Eqs. (24) and (25) are given by

$$f_0 = 1 + C_3 e^{-\lambda_1 \eta} + B_1 e^{-m_1 \eta} + C_1 e^{-m_1 y} \tag{25}$$

$$f_1 = 1 + C_4 e^{-\lambda_3 \eta} + B_3 e^{-m_3 \eta} + D_3 e^{-m_1 \eta} + E_3 e^{-\lambda_2 \eta} + F_3 e^{-Sc \eta} + G_3 e^{-\lambda_1 \eta} \tag{26}$$

$$g_0 = e^{-m_1 \eta} \tag{28}$$

$$g_1 = (1 - A_2) e^{-m_3 \eta} + A_2 e^{-m_1 \eta} \tag{29}$$

$$h_0 = e^{-Sc \eta} \tag{30}$$

$$h_1 = e^{-\lambda_2 \eta} + ASc\left(e^{-\lambda_2 \eta} - e^{-Sc \eta}\right)/n \tag{31}$$

The constants appeared in the equations (26) to (31) are not presented here due to shake of brevity.

The skin-friction coefficient is a important physical parameter for this type of boundary-layer flow. This parameter can be defined and determined as follows:

$$C_f = \frac{\tau_w}{\rho U_0 v_0} = \left.\frac{\partial u}{\partial y}\right|_{y=0} = 1 - \lambda_1 C_3 - B_1 m_1 - Sc C_1$$

$$+\varepsilon e^{nt}\left(1 - \lambda_3 C_4 - B_3 m_3 - m_1 D_1 - \lambda_2 E_3 - Sc F_3 - \lambda_1 G_3\right)$$

It should be mentioned that in the absence of the concentration buoyancy and heat absorption effects, all of the flow and heat transfer solutions reported above are consistent with those reported earlier by Kim [19].

Results and Discussion

Numerical evaluation of the analytical results reported in the previous section was performed and a representative set of results is reported graphically in Figs. 8.1–8.5 for various values of A=1.5, G_T=5.0, G_c=5.0, K = 0.1, M=2, n=0.5, Pr = 0.71, U_P=1, ε=0.01, ϕ=1.0, t=0.5, Sc=0.6. These results are obtained to illustrate the influence of the solutal Grashoff number G_c, the heat absorption coefficient f and the Schmidt number Sc on the velocity, temperature and the concentration profiles. The effects of the other physical parameters on the solutions were reported previously by Kim [19] and, therefore, will not be repeated herein.

Fig. 8.1 presents typical velocity profiles in the boundary layer for

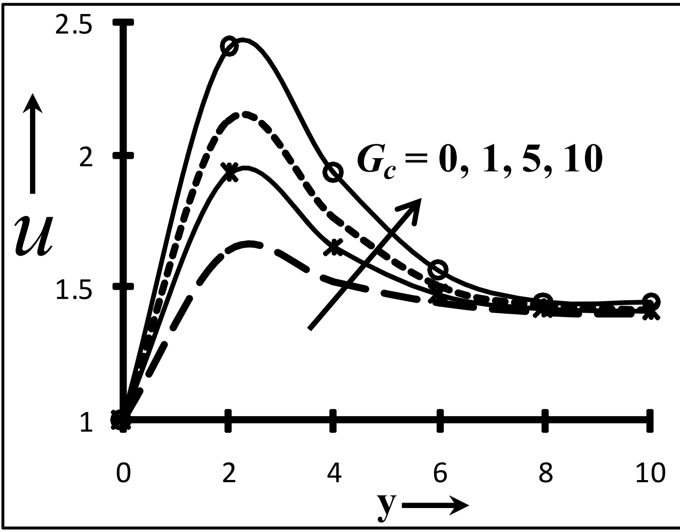


Fig. 8.1 Velocity profiles for Gc against y

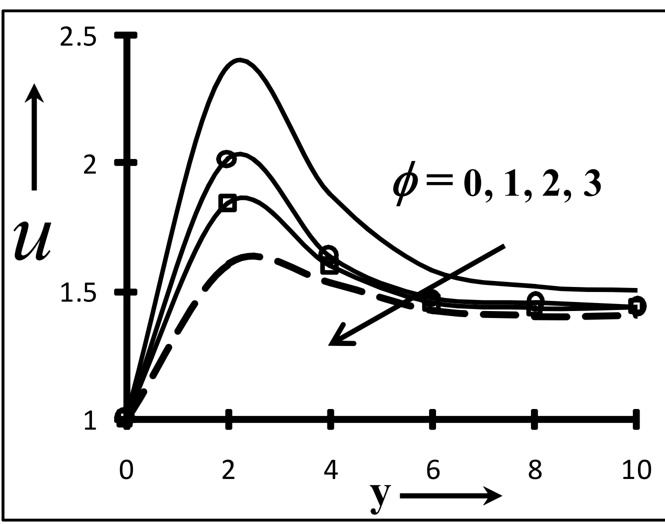


Fig. 8.2 Velocity profiles for ϕ against y

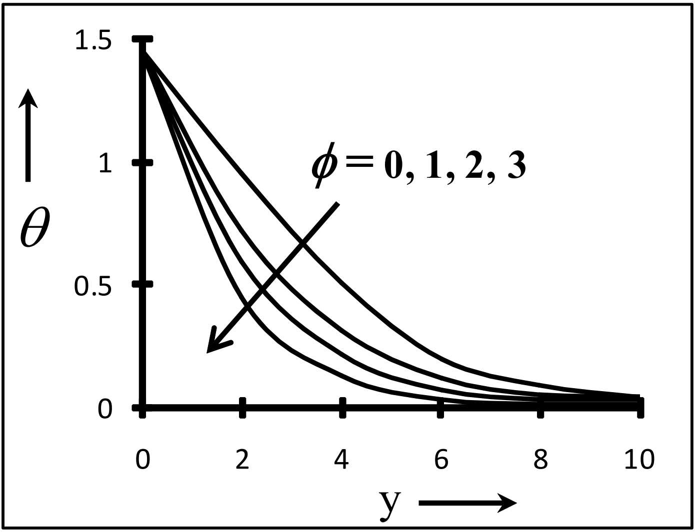


Fig. 8.3 Temperature profiles for ϕ against y

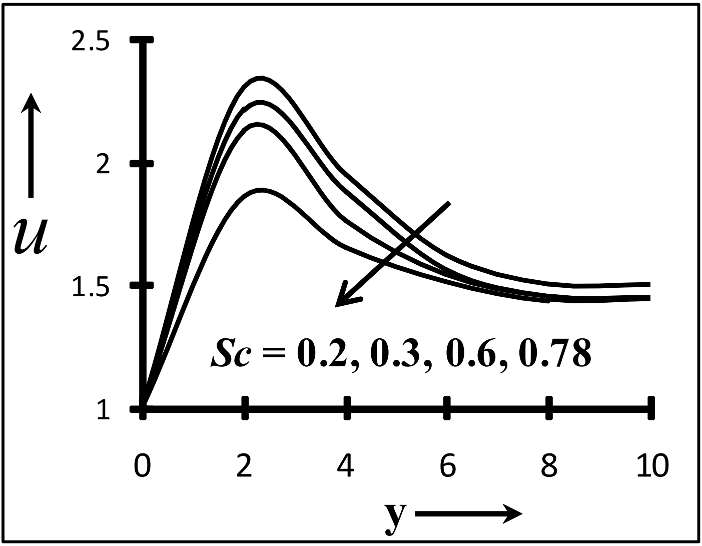


Fig. 8.4 Velocity profiles for *Sc* against y

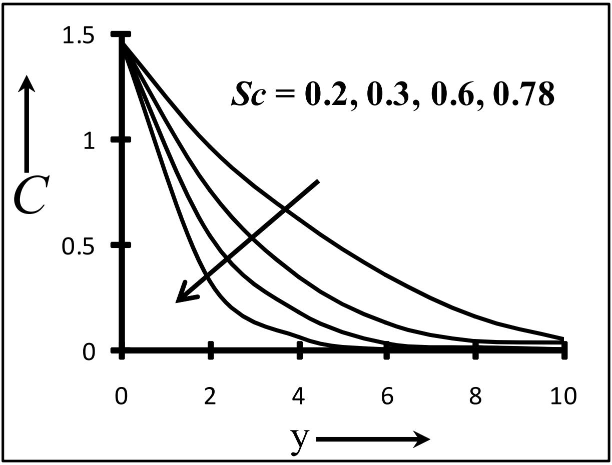


Fig. 8.5 Concentration profiles C and Sc against y:

Table 1: Effects of G_c on C_f for the reference values in Fig. 1

Table 2: Effects of ϕ on C_f for the reference values in Fig. 2

Table 1		Table 2	
G_c	C_f	ϕ	C_f
0	2.7200	0	4.5092
1	3.2772	1	4.2167
5	4.8343	2	4.1091
10	7.1910	3	4.0785
15	12.2433	4	3.9108

various values of the solutal Grashoff number G_c while all other parameters are kept at some fixed values. The velocity distribution attains a distinctive maximum value in the vicinity of the plate surface and then decreases properly to approach the free stream value. As expected, the fluid velocity increases and the peak value becomes more distinctive due to increases in the concentration buoyancy effects represented by G_c. This is evident in the increases of U as G_c increases in Fig. 8.1.

Figs. 8.2 and 8.3 illustrate the influence of the heat absorption coefficient f on the velocity and temperature profiles, respectively. Physically speaking, the presence of heat absorption (thermal sink) effects has the tendency to reduce the fluid temperature. This causes the thermal buoyancy effects to decrease resulting in a net reduction in the fluid velocity. These behaviors are clearly obvious from Figs. 8.2 and 8.3 in which both the velocity and temperature distributions decrease as f increases. It is also observed that both the hydrodynamic (velocity) and the thermal (temperature) boundary layers decrease as the heat absorption effects increase.

Figs. 8.4 and 8.5 display the effects of the Schmidt number Sc on the velocity and concentration profiles, respectively. As the Schmidt number increases, the concentration decreases. This causes the concentration buoyancy effects to decrease yielding a reduction in the fluid velocity. The reductions in the velocity and concentration profiles are accompanied by simultaneous reductions in the velocity and concentration boundary layers. These behaviors are clearly shown in Figs. 8.4 and 8.5.

Tables 1–2 depict the effects of the solutal Grashof number G_c for the reference values in Fig. 1, and the heat absorption coefficient f for the reference values in Fig. 2 on the skin-friction coefficient C_f, respectively. It is observed from these tables that as G_c increases, the skin friction coefficient increases. However, as the heat absorption effects increase the skin-friction.

Conclusions

The governing equations for unsteady MHD convective heat and mass transfer past a semiinfinite vertical permeable moving plate embedded in a porous medium with heat absorption was formulated. The plate velocity was maintained at a constant value and the flow was subjected to a transverse magnetic field. The resulting partial differential equations were transformed into a set of ordinary differential equations using two-term series and solved in closed-form. Numerical evaluations of the closed-form results were performed and some graphical results were obtained to illustrate the details of the flow and heat and mass transfer characteristics and their dependence on some of the physical parameters. It was found that when the solutal Grashof number increased, the concentration buoyancy effects were enhanced and thus, the fluid velocity increased. However, the presence of heat absorption effects caused reductions in the fluid temperature which resulted in decreases in the fluid velocity. Also, when the Schmidt number was increased, the concentration level was decreased resulting in a decreased fluid velocity. In addition, it was found that the skin friction coefficient increased due to increases in the concentration buoyancy effects while it decreased due to increases in either of the heat absorption coefficient or the Schmidt number. However, the Nusselt number decreased as the heat absorption coefficient was increased and the Sherwood number decreased as the Schmidt number was increased.

References

1. Chen, T. S., and Strobel, F. A., Buoyancy effects in boundary layer adjacent to a continuous, moving horizontal flat plate, *Trans ASME J Heat Transf,* **102**, 170–2 (1980).
2. Soundalgekar, V. M., Viscous dissipation effects on unsteady free convection flow past an infinite, vertical porous plate with constant suction, *Int J Heat Mass Transf*, **15**, 1253–61 (1972).
3. Abdelkhalek, M. M., Heat and mass transfer in MHD free convection from a moving permeable vertical surface by a perturbation technique, *Commun Nonlinear Sci Numer Simul*, **14**(5), 2091–2102 (2009).
4. Sharma, P. K., and Chaudhury, R. C., Effect of variable suction on transient flow convective viscous incompressible flow past a vertical plate with periodic temperature variations in slip flow regime, *Emirates J Eng Sci*, **8**, 33–8 (2003).
5. Ahmed, S., Influence of chemical reaction on transient MHD free convection flow over a vertical plate in slip-flow regime, *Emirates J Eng Res*, **15**(1), 25–34 (2010).
6. Hayat, T., Nadeem, S., Pudasaini, S. P., and Asghar, S., Fluctuating flow of a third-order fluid past an infinite plate with variable suction, *Arch Mech*, **55**(3), 305–24 (2003).
7. Muthucumarswamy, R., and Kumar, S. G., Heat and mass transfer effects on moving vertical plate in the presence of thermal radiation, *Theoret Appl Mech*, **31**, 35–46 (2004).
8. El-Hakim, M. A., MHD oscillatory flow on force-convection radiation through a porous medium with constant velocity, *J Mag Magnetic Mater*, **220**, 271–6 (2000).
9. Ahmed, S., The study of heat and mass transfer on free convective three-dimensional unsteady flows over a porous vertical plate, *Journal*

of Energy, Heat and Mass Transfer, **31**, 89-110 (2009).

10. Ahmed, S., and Liu, I-Chung, Mixed convective three-dimensional heat and mass transfer flow with transversely periodic suction velocity, *Int. J. Applied Mathematics and Mechanics*, **6**, 58-73 (2010).
11. Ahmed, S., Free and forced convective MHD oscillatory flows over an infinite porous surface in an oscillating free stream, *Latin American J. Applied Research*, **40**, 167-173 (2010).
12. Chamkha, A. J., Thermal radiation and buoyancy effects on hydrodynamic flow over an accelerated permeable surface with heat source and sink, *Int. J. Engineering Science*, **38** (15), 1699-1712 (2000).
13. Lai, F. C., and Kulacki, F. A., *Int. J. Heat Mass Transfer*, **113**, 252 (1991).
14. Chamkha, A. J., Unsteady hydromagnetic natural convection in a fluid-saturated porous medium channel, *Advanced Filtration and Separation Technology*, **10**, 369–375 (1996).
15. B´eg, O. A., Takhar, H. S., and Singh, A. K., Multi-parameter perturbation analysis of unsteady oscillatory magneto-convection in porous media with heat source effects, *Int. J. Fluid Mechanics Research*, **32**(6), 635–661 (2005).
16. Ahmed, S. and Kalita, K., Magnetohydrodynamic transient flow through a porous medium bounded by a hot vertical plate in presence of radiation: A theoretical analysis, *J. Engineering Physics and Thermophysics*, **86**(1), 31-39 (2013).
17. Cramer, K. R., and Pai, S. I., Magnetofluid Dynamics for Engineers and Applied Physicists, McGraw-Hill, New York, 1973.
18. Kim, Y.J., Unsteady MHD convective heat transfer past a semi-inûnite vertical porous moving plate with variable suction, *Int. J. Eng. Sci*., **38**, 833 (2000).

9

A Study on Curricula Aspect of Mathematics in 10+2 Standard: An Ex Post Facto Research Study in Goalpara District of Assam

Abdul Wahed

Abstract

This study deals with the curriculum aspect of mathematics in +2 standard in Arts, Science and commerce stream of Assam. The knowledge of mathematics is mandatory and utmost required for all the branches of learning. But the amount of knowledge of mathematics and the branches of mathematics to be taught for the necessity of different streams should not be same. Attempt was made to find out the answer of an important question whether the syllabus of mathematics in +2 standard is feasible for all the streams. The investigator selects purposively Goalpara District of Assam under the study. An opinion poll was conducted regarding the views on the present mathematics syllabus of +2 standard in the study area among the mathematics teachers of Higher Secondary Schools and Colleges. This study reveals that majority teachers opined for change in the curriculum so that it become viable for all the streams and the status of mathematics education can be improved in all aspects.

Introduction

Mathematics enables us to interpret and manipulate the ideas in all branches of learning as well as in all our daily activities from dawn to midnight (Betsur, 2005). The importance of mathematical learning has repeatedly been emphasized by the educators (Arunachalam.2001; Wilkins & Ma, 2002; Lall, 2005; Noyas, 2011.) and in India National Report, 2008. But now a day, it has been a matter of grave concerned for Mathematics stake-holders that the level of achievement and commitment of the students to learn Mathematics is reducing day by day. In many countries there are short fall of the qualified person with sound mathematical knowledge as required by industry and business sector of the countries (Report of the Royal Society, 2008; Kounine et.al. 2008). Moreover, in a study of international comparison of post-16 mathematics education in 24 different countries Hodgen et. al. (2010) revealed that England, Wales and Northern Ireland have the lowest levels of participation in upper secondary mathematics. This has a great impact in modern science and technology as well as in the field of economy of a country. So, an in depth study is felt necessary to investigate the status of mathematics curricula in Assam in upper secondary level which is the base of higher education.

Mathematics is deeply involved with all the issues affecting society (Ambrosio, 2003). In view of the importance of mathematics and considering its deteriorating situation many researchers (Varadarajan, 1983; James, et.al. 2012) pointed out that the curriculum of mathematics should be reframed keeping it related to everyday life. So that it has got wide applications and professional opportunities for the young generation. In the new curriculum there should be a different philosophy and set of values, so as to fulfill the need, hope and aspirations of the majority students (Noyas, 2007).

The choice of the topic is disposed on the current world trend and research emphasis on the curricula aspect in post-16 mathematics in different countries. Hodgen et. al. (2010), in a report of Nuffield Foundation, suggested offering different types and level of mathematics in post-16 curriculum within the context of current and projected social and economic needs. They also pointed out that many countries such as Germany, Hong Kong and New Zealand have already been introduced compulsory basic mathematics into upper secondary level. But they introduced separate optional papers for basic and advanced course to fulfill the socio-economic necessity and demand for STEM (science, technology, engineering and mathematics). It should be noteworthy to mention herein that the national curriculum of upper secondary mathematics in China has been introduced separate optional papers for humanities group, science and technology, advance level mathematics in pure and applied groups. It consists of 5 required modules (compulsory) and 4 optional series.

In the context of India, NCERT has been taking the responsibility of developing the school curriculum science 1975. In 2000, the National Curriculum Framework for School Education (NCFSE) was brought out to ensure the prevailing socio-economic needs. It was again reviewed in 2005 with a view to respond to the new developments and necessity. The National Curriculum Framework (NCF 2005) was developed by NCRT with the help of National Steering Committee and the position papers prepared by twenty-one Focus Groups. After that NCERT developed the Curriculum, syllabus and textbooks for schools in the light of the recommendations of NCF. The NCF-2005 was approved by Central Advisory Board of Education (CABE). The states and union territories either adopt or adapt the school curriculum and textbooks developed by NCERT. No doubt, the efforts and sincerity of NCERT to develop a better curriculum for school education is appreciable. But the change and reform made in the national curriculum for mathematics in higher secondary standard is not adequate to all of its measures. Moreover, the vision of the NCERT "all students can learn mathematics and that all students need to learn mathematics" is completely spoiled with the prevailing very low rate (8.15%) of participation in the subject in higher secondary standard. This is due to the fact that the curriculum of mathematics is not attractive for arts students and they have found no opportunities and advantage for their future perspectives. The present curriculum has been developed giving priority and importance to the need of the science students only and ignoring the necessity of others (Wahed, et.al.2013). It has been a matter of grave concerned that the curriculum of higher secondary mathematics is completely biased in the sense that there is no provision for separate papers for arts, science, commerce and vocational streams. But on the contrary many developed and developing countries have already been introduced two types of papers compulsory and optional in the curriculum of upper secondary mathematics. So to cope up India in general and in particular Assam with modern trend of mathematics education an in depth study is felt necessary in Goalpara District, a small part of the country.

Research Questions

In relation to the curriculums of mathematics in +2 standard of arts, science & commerce streams of Assam, this study proposes to address the following three important research questions.

1. What is the modern trend of mathematics education in developed countries?
2. Is the existing syllabus of Assam Higher Secondary Education Council in mathematics feasible for all the streams?
3. Are there any need of separate syllabus for arts, science & commerce streams in mathematics?

Methodology:

There are 9 institutions (5 colleges and 4 higher secondary schools) offering the study programme of mathematics in +2 level in Goalpara District of Assam. Moreover, there are 173 schools (Secondary & higher secondary standard) in the district offering compulsory mathematics programme up to 10^{th} class. All these 182 institutions constitute the population for the present study. Stratified sampling technique is applied for the study. At first 20 [Prof.& PGT] mathematics teachers are selected from the institutions of +2 standard. Besides these another 20 mathematics teachers are selected randomly from 161 graduate mathematics teachers working in the district. Thus, the sample size taken is 40 number of mathematics teachers and they were provided with a pre tested teachers' questionnaire. From these 40 respondents primary information are gathered regarding their views in the mathematics curricula of +2 standard. Secondary information regarding participation and achievement in mathematics in +2 level are gathered from official records as provided by the concerning authorities of the 9 institutions. The necessary information regarding the curricula of foreign countries are collected from the proper web pages. This study is descriptive in nature and simple statistical tools are applied to analyze the data.

A brief Review on the curricula of developed nations

To justify the feasibility of the mathematics syllabus in +2 standard in Assam it is necessary to look into the prevailing curricula of different countries around the world. Hodgen, et.al. (2013), in their study "towards universal participation in post-16 mathematics: lessons from high-performing countries" forwarded from Nuffield Foundation highlighted the modern trend of mathematics education of a few developed countries in higher secondary level as presented herein below.

1. England:

a) Schooling is compulsory up to the age of 16 but this will increase to age 17 by 2013 and age 18 by 2015. Mathematics is also compulsory up to age 16.

b) Upper secondary stage begins after age 16 and post-16 mathematics is not compulsory, but students who choose this subject have options including AS or A level in mathematics.

c) Participation rate in any mathematics at post-16 level are very low (20%) in comparison to other developed countries.

d) There is wide range of vocational qualifications in post-16 education which includes mathematics in different level and standard depending on the course.

2. Germany:

a) Upper secondary stage begins after age 16 and 20% students received general education where as 80% students take various courses of vocational education.

b) Mathematics is compulsory on almost all general and vocational courses. But the standard of mathematics course varies between a basic or advanced course (in general education) and the types of vocational courses.

c) More than 90% students participate in post-16 mathematics.

3. New Zealand:

a) Schooling is compulsory up to the age of 16 and there are no compulsory subjects at senior secondary education.

b) Two distinct options are available in advanced mathematics course. One is mathematics with calculus and other is mathematics with statistics. Moreover students can choose to take a greater or smaller amount of mathematics in each option and can choose to take modules from both the options.

c) Participation rate in post-16 mathematics are very high (71%)

4. Singapore:

a) Upper secondary education begins at the age 16 and there are no compulsory subjects at post-16 education.

b) Arts and humanities students must take a science or mathematics option.

c) A levels students are required to take a contrasting subject as a result 80% students take some mathematics course at post-16 level.

5. China:

Wang Linquan, in his study "what challenges are we confronted within high school mathematic" upheld the issue about the fundamental structure of national curriculum standards of post-16 mathematics in china and he highlighted the curriculum as follows-

a) There are 5 required modules and 4 optional series in the post-16 curriculum of mathematics. The 4 optional series are limited optional series-1, limited optional series-2, freely optional series-3 and freely optional series-4.

b) Limited optional series-1 is provided for the students who are preparing for further study of social sciences and it consists of two groups.

c) Limited optional series-2 is provided for the students who prepare for further study of science and technology and it consist of three groups.

d) Freely optional series-3 includes 6 topics of introduction of modern pure mathematics ideas.

e) Freely optional series 4 includes 10 topics of basic methodologies of applied mathematics.

The above discussion makes it clear that the countries like England, New Zealand, Germany and China are providing separate optional papers in mathematics in post-16 curriculum. The picture of Singapore is quite different as the students of arts and humanities group must take either a science or mathematics option in upper secondary classes.

Analysis:

Feasibility of the Curriculum:

The percentage of participation and achievement in mathematics are found year wise separately for arts, science and commerce streams. Then their averages for10 years [from 2002 to 2011] are calculated and presented in the following bar diagram-1 & 2 for direct comparison. From the diagram it is evident that participation rate in mathematics is very low (diag-1) in arts and commerce streams but those who participate in the programme they can make achievements (diag-2) in the subject. On the contrary, the participation rate in mathematics is very high (diag-1) in science stream but their achievements are not so satisfactory (diag-2).

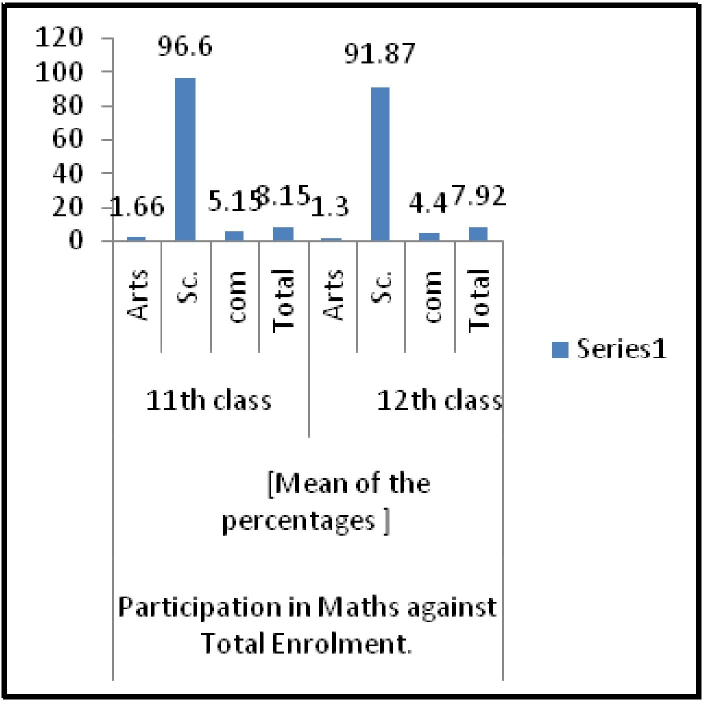


Diagram-1

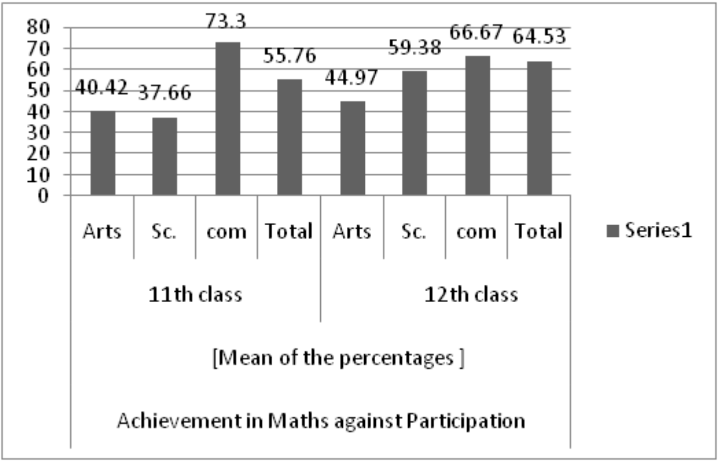


Diagram-2

Opinion poll on the Curriculum

The result of the opinion poll of 40 mathematics teachers regarding the curriculum of mathematics in higher secondary level are shown in the following bar diagram-3. From the diagram it is clear that majority of the teachers opined that the prevailing syllabus of mathematics is not feasible for all the streams and there should be separate syllabus for arts, science and commerce streams.

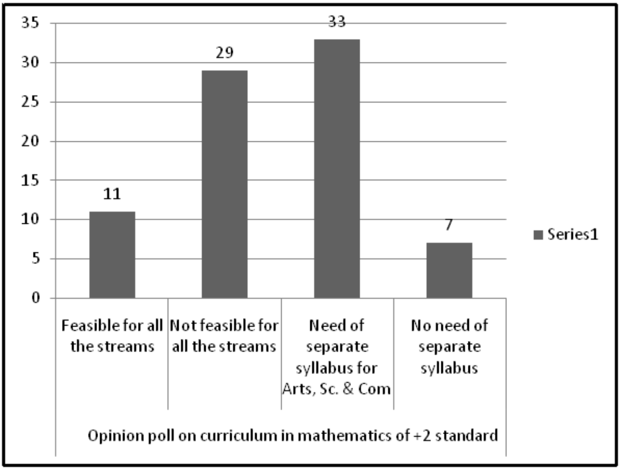


Diagram-3

Findings of the study

The findings of the study are summarized as follows-

1. The rate of participation in mathematics in higher secondary level are found very low in Goalpara District.
2. Majority of the teachers consider that the curriculum of Assam Higher Secondary Education Council in mathematics is not feasible for all the streams and they voted for separate syllabus in arts, science and commerce streams.

Recommendations of the Study

1. The new curriculum of upper secondary mathematics should be developed in order to ensure that a large majority of young people can participate in the study programme of mathematics.
2. The new curriculum should introduce different types and level of optional mathematics within the context of current and projected socio- economic need so that all young people can be placed to benefit from their studies in mathematics.
3. The new curriculum should be given more prominence in the knowledge and experience of computer software as an essential skill in employment and higher education so that it can attract more students.

Conclusions:

The rate of participation in mathematics in arts (1.66%) and commerce (5.15%) stream indicate that the curriculum of mathematics is not so attractive to them. On the other hand in the science stream 96.6% students participate in the study programmme of mathematics. That means the prevailing curriculum of mathematics is completely science stream oriented. Therefore, the basic problem is how to make the curriculum of mathematics acceptable to all the streams in order to increase the rate of participation in the subject. However, the study has its own limitation as it covers only a small part of the country and reframing of curriculum is a national agenda. Therefore, to arrive at a consensus on reframing of mathematics curriculum it requires further details study and research covering the whole state and nation.

References

1. Royal Society. Science and mathematics education 14-19: A 'state of the nation' report on the participation and attainment of 14-19 years old in science and mathematics in the UK, London: p.17.Fromhttp://royalsociety.org/uploadedFiels/Royal_society_content/influencing_policy/Education/R. (2008)
2. Kounine, Marks, & Truss. The Value of Mathematics. London: Reform: p.5. From, www.reform.co.uk/content/4504/research/education/the_value_of_mathematics. (2008)
3. Hodgen, J., Pepper, D., Sturman, L.,and Ruddock,G., Is the UK an outlier? An international comparison of upper secondary mathematics education: Nuffield Foundation. From www.nuffieldfoundation.org/sites/default/files/fiels/Is the UK an outlier Nuffield Foundation_v_FINAL.pdf. (2010)
4. Noyes, A., Participation in mathematics: What is the problem?" *Improving Schools*, *SAGE journals online*, July 14 (2):p.12, from http://imp.sagepub.com/content/12/3/277.full.pdf+html. (2011)
5. Hodgen, J., Marks, R., & Pepper, D., Towards universal participation in post-16 mathematics: lessons from high-performing countries: Nuffield Foundation. From www.nuffieldfoundation.org/sites/default/files/fiels/towards_universal_participation_in_post_16_maths_v_FINAL.pdf. (2013)
6. D'Ambrosio,U., The role of mathematics in building up a democratic society. http://www.maa.org/ql/qltoc.html. (2003)
7. Varadrajan, V. S., Mathematics in and out of Indian Universities. The Mathematical Intelligencer, **5**, 38-42 (1983).
8. James,A. & Balasubramaniam,P.S., Teaching of Mathematics. Reprint of First Edition, Neelkamal Publications Pvt. Ltd. New Delhi (2012).
9. Noyes, A., Rethinking *school mathematics*, London, paul Chapman publishing, (2007).
10. Wilkins, J.L., and Ma, X., Predicting students' growth in mathematical content knowledge. *The Journal of Educational Research*, **95**, 288-298 (2002).
11. Arunachalam, S., Mathematics Research in India today: What does the literature reveal? *Scientometrics,* **52** (2), 235-259, (2001).
12. Betsur, C. Ningamma (2005), Reasoning strategies in Mathematics, New Delhi, Anmol Press. Pvt. Ltd.
13. Lall, M (2005). The Challenges for India's Education System, Chatham House, Asia Programme.
14. Status of Education in India National Report (2008). National University

of Educational Planning & Administration, Department of Higher Education, MHRD, Govt. of India.

15. NCERT (2006), position paper -1.2, National Focus Group on Teaching of Mathematics. http://www.ncert.nic.in

16. Wang Linquan, What Challenges Are We Confronted With in High School Mathematics? http://dg.icme11.org/document/get/118.

10

Hydromagnetic Convective Radiating Laminar Flow past an infinite Vertical Porous Plate immersed in a time dependent Porous medium with Heat Source

Bharati Partin
Sahin Ahmed

Abstract

Mathematical investigation of Laminar free convection flow of an incompressible, electrically conducting, viscous, radiating liquid through time dependent porous medium past an infinite, isothermal, vertical, porous plate is presented. A constant heat source is considered in the flow region in the presence of transversely applied uniform magnetic field and a time dependent suction velocity. Perturbation technique has been employed to obtain the analytical solutions of velocity field and temperature field. *Rosseland approximation* is used for the radiation flux. The expressions for skin friction and rate of heat transfer are also obtained. It is observed that, the flow velocity is decelerated with the increasing effect of Magnetic parameter, but this trend is reversed in case of permeability parameter. Also temperature field is observed to be reduced under the influence of radiation parameter and graphs are presented graphically.

Keywords: *Viscous liquid; Porous medium; Magnetic field; Convective Flow; Time dependent suction velocity; Heat source/sink.*

Introduction

Theoretical analysis of convective motion and heat transfer in a porous medium are essential for understanding many engineering problems including fixed bed reactors, the solidification of alloys, convection over heat exchanger tubes, solar heating systems, thermal insulations, packed distillation columns, absorbent beds and applications in ground water hydrology, etc. Recent monographs by Nield and Bejan [1] and Ingham and Pop [2] provided an excellent summary of the work on the subject. In the case of large temperature differences between the surface and the ambient causes, the radiation effect may become important in natural convection, as in many engineering processes. Raptis and Perdikis [4] studied the effects of thermal radiation on moving vertical plate in the presence of mass diffusion. Jaiswal and Soundalgekar [7] obtained an approximate solution to the problem of an unsteady flow past an infinite vertical plate with constant suction and embedded in a porous medium with oscillating plate temperature. An analysis of the thermal radiation effects on stationary mixed convection from vertical surfaces in saturated porous media for both a hot and a cold surface has been presented by Bakier [8]. Sahin [12] investigated the effect of transverse periodic permeability oscillating with time on the heat transfer flow of a viscous incompressible fluid through a highly porous medium bounded by an infinite vertical porous plate, by means of series solution method. Sahin [13] studied the effect of transverse periodic permeability oscillating with time on the free convective heat transfer flow of a viscous incompressible fluid through a highly porous medium bounded by an infinite vertical porous plate subjected to a periodic suction velocity. Sahin and Chamkha [16] analyzed the effects of radiation and chemical reaction on steady mixed convective heat and mass transfer flow of an optically thin gray gas over an infinite vertical porous plate with constant suction in presence of transverse magnetic field. Also, Sahin and Liu [18] analyzed the effects of mixed convection and mass transfer of three-dimensional oscillatory flow of a viscous incompressible fluid past an infinite vertical porous plate in presence of transverse sinusoidal suction velocity oscillating with time and a constant free stream velocity. Makinde [20] used a superposition technique and a Rosseland diffusion flux model to study the natural convection heat and mass transfer in a gray, absorbing–emitting fluid along a porous vertical translating plate.

The aim of the present study is to investigate the time dependent porosity of the medium as well as the radiation effects on hydro magnetic convective flow of viscous liquid past an infinite vertical plate embedded in a porous medium in the presence of a heat source by a *perturbation* scheme.

Formulation of the problem

In Cartesian system, we consider the two dimensional flow of an electrically conducting incompressible viscous liquid through a porous medium past an infinite isothermal vertical porous plate with constant heat source and radiation at y = 0 in the presence of a uniform magnetic field B_o applied normal to the flow. The porous medium is a non-homogeneous. Therefore, it can be taken as a function of *t* say *k*(*t*). The analysis to the present problem is based on the following assumptions:

1. The x-axis is along the plate y-axis normal to it.
2. A uniform magnetic filed is applied and acts in the y-direction.
3. The permeability of the medium is an exponential decreasing function of time.
4. The plate temperature varies is the presence of a heat source.
5. The magnetic Reynold's number is small so the induced magnetic field is negligible.
6. The plate is infinite in extent; therefore, all the physical variables except pressure depend on *y* and *t* only.
7. The suction velocity is an exponential decreasing function of time.

The governing equations for continuity, momentum of energy for the present problem under the present configuration are:

$$\frac{\partial V}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \mathrm{v}\,\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \mathrm{v}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\sigma B_0^2}{\rho} + \frac{v}{K(t)}\right)u \tag{2}$$

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial y} = \frac{K_T}{\rho C_P}\frac{\partial^2 T}{\partial y^2} - \frac{S}{\rho C_P}(T - T_\infty) - \frac{1}{\rho C_P}\frac{\partial q_r}{\partial y} \tag{3}$$

where *r* is the density, *g* is the acceleration due to gravity, *b* is the coefficient of volume expansion, í is the Kinematic viscosity, T_∞ is the temperature of the fluid in the free stream, ó is the electric conductivity,

B_0 is the magnetic induction, *S* is the source/sink coefficient, q_r is the radiative heat flux.

By using *Rosseland approximation* for the radiation we take

$$q_r = -\frac{4\sigma^*}{3K^*}\frac{\partial T^4}{\partial y} \tag{4}$$

where ó* the Stefan - Boltzmann constant and k* the mean absorption coefficient.

We assume that the temperature differences within the flow are such that T^4 may be expressed as a linear function of temperature.

This is accomplished by expanding T^4 in a Taylor series about T_∞ and neglecting higher-order terms, thus

$$T^4 = 4T_\infty^3 T - 3T_\infty^4 \tag{5}$$

The continuity equation (1) shows that v is a function of time only. Therefore, we assume:

$$V = -V_0(1 + \varepsilon e^{-nt}) \tag{6}$$

where $V_0 > 0$ is a real constant, *n* is a positive constant, *å* is a small quantity (<<1) and the negative sign indicates that the suction is towards the plate.

Using assumption (6) we assume:

$$K(t) = K_0\ (1 + \varepsilon e^{-nt}) \tag{7}$$

where K_0 is the constant permeability of the medium.

The boundary conditions for the present problem are:

$$u = U_0\,(1 + \varepsilon e^{-nt}), \qquad T = T_\infty\,(1 + \varepsilon e^{-nt})\ at\ y = 0$$

$$u \to 0, \qquad T \to T_\infty\ as\ y \to \infty \tag{8}$$

Substituting (6) and (7) in (2) and (3), we get :

$$\frac{\partial u}{\partial t} - V_0\,(1 + \varepsilon e^{-nt})\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + V\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho} - \frac{\nu u}{K_0\,(1 + \varepsilon e^{-nt})} \tag{9}$$

$$\frac{\partial T}{\partial t} - V_0 \left(1 + \varepsilon e^{-nt}\right) \frac{\partial T}{\partial t} = \frac{K_T}{\rho C_P} \frac{\partial^2 T}{\partial y^2} - \frac{S}{\rho C_P}(T - T_\infty) - \frac{1}{\rho C_P} \frac{\partial q_r}{\partial y} \quad (10)$$

On introducing the following non dimensional quantities

$$u^* = \frac{u}{U_0}, t^* = \frac{tV_0^2}{v}, y^* = V_0 \frac{y}{v}, T^* = \frac{T - T_\infty}{v}, n^* = \frac{nv}{V_0^2}$$

Equations (9) and (10) after dropping the asterisks (*) can be written as

$$\frac{\partial u}{\partial t} - \left(1 + \varepsilon e^{-nt}\right) \frac{\partial u}{\partial y} = G_r T + \frac{\partial^2 u}{\partial y^2} - M^2 u - \frac{u}{K_0\left(1 + \varepsilon e^{-nt}\right)} \quad (11)$$

$$\frac{\partial T}{\partial t} - \left(1 + \varepsilon e^{-nt}\right) \frac{\partial T}{\partial y} = \frac{1}{P_r}\left(1 + \frac{4N}{3}\right) \frac{\partial^2 T}{\partial y^2} - ST \quad (12)$$

with the boundary conditions :

$$u = \left(1 + \varepsilon e^{-nt}\right), T = \left(1 + \varepsilon e^{-nt}\right) \quad at\ y = 0$$

$$u \to 0, \qquad T \to 0 \quad as \quad y \to \infty \quad (13)$$

$K_0^* = \frac{K_0 V_0^2}{v^2}$ (Permeability parameter),

$S^* = \frac{vS}{\rho c_P V_0^2}$ (Heat Source parameter),

$M = \sqrt{\frac{\sigma B_0^2 v}{\rho V_0^2}}$ (Hartmann number),

$Gr = \frac{gBv^2}{U_0 V_0^2}$ (Grashoff number),

$Pr = \frac{\mu C_p}{K_T}$ (Prandtl number),

$N = \left(\frac{4\sigma^* T_\infty^3}{k^* K_T}\right)$ (Radiation parameter),

Solution of the Problems

The solutions of equations (11) and (12) are:

$$u = u_0 + \varepsilon u_1 e^{-nt} \text{ and } T = T_0 + \varepsilon T_1 e^{-nt} \tag{14}$$

Substituting (14) in the equations (11) and (12) and comparing the harmonic and non harmonic terms, we get:

$$u_0'' + u_0' - M_1 u_0 = -G_r T_0 \tag{15}$$

$$u_1'' + u_1' - M_2 u_1 = -G_r T_1 \tag{16}$$

$$\left(1 + \frac{4N}{3}\right) T_0'' + P_r T_0' - S P_r T_0 = 0 \tag{17}$$

$$\left(1 + \frac{4N}{3}\right) T_1'' + P_r T_1' - P_r S_1 T_1 = -P_r T_0' \tag{18}$$

where $M_1 = M^2 + \frac{1}{K_0}$, $\quad M_2 = M^2 + \frac{1}{K_0} - n$, $\quad S_1 = S - n$

With the boundary conditions

$$u_0 = 1, u_1 = 1, T_0 = 1, T_1 = 1 \;\; at\; y = 0$$

$$u_0 \to 0, u_1 \to 0, T_0 \to 0, T_1 \to 0 \;\; as\; y \to \infty \tag{19}$$

The solution of the above coupled equation (15) to (18) under the boundary condition (19), we get:

$$u = (1 + a_2) e^{-m_3 y} - a_2 e^{-m_1 y}$$

$$+\varepsilon[(1 - a_6 - a_7 - a_8) e^{-m_4 y} + a_6 e^{-m_3 y} + a_7 e^{-m_2 y} + a_8 e^{-m_1 y}] e^{-nt} \tag{20}$$

$$T = e^{-m_1 y} + \varepsilon[(1 + a_1) e^{-m_2 y} - a_1 e^{-m_1 y}] e^{-nt} \tag{21}$$

The skin friction coefficient (t) at the plate at y = 0 is:

$$\tau = -\left(\frac{\partial u}{\partial y}\right)_{y=0} = m_1 A_2 - (1 + A_2) m_3$$

$$+\varepsilon[(A_6 + A_7 + A_8 - 1) m_4 - A_6 m_3 - A_7 m_2 - A_8 m_1] e^{-nt} \tag{22}$$

The rate of heat transfer in terms of Nusselt number (N_u) at the plate at y = 0 is:

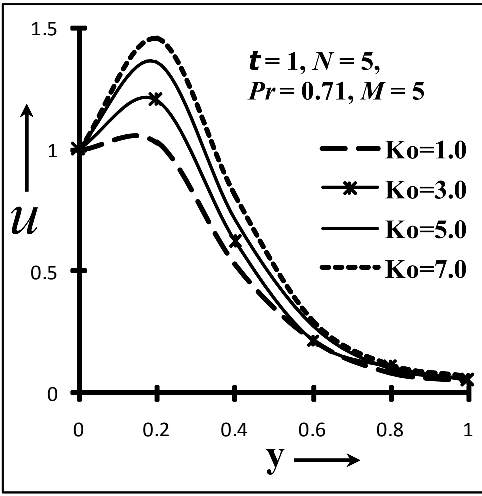


Fig. 10.1 Velocity field for the variations of porosity against y

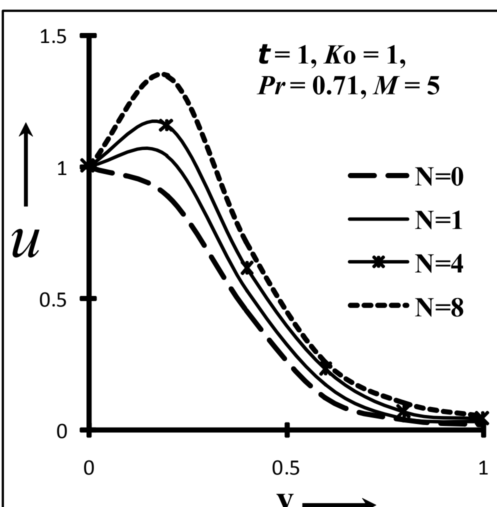


Fig. 10.2 Velocity field for the variations of radiation against y

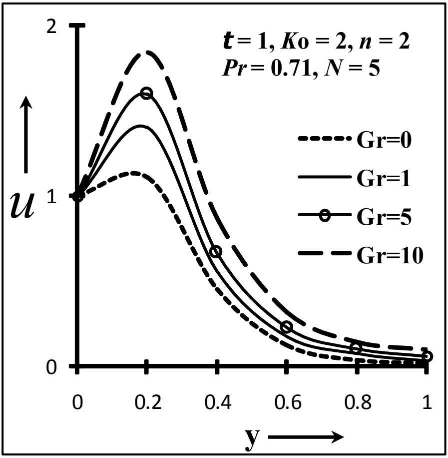


Fig. 10.3: Velocity field for the variations of Grashoff number against y

$$N_u = -\left(\frac{\partial T}{\partial y}\right)_{y=0} = m_1 + \varepsilon[m_1 a_1 - (1 + a_1)m_2]e^{-nt} \qquad (23)$$

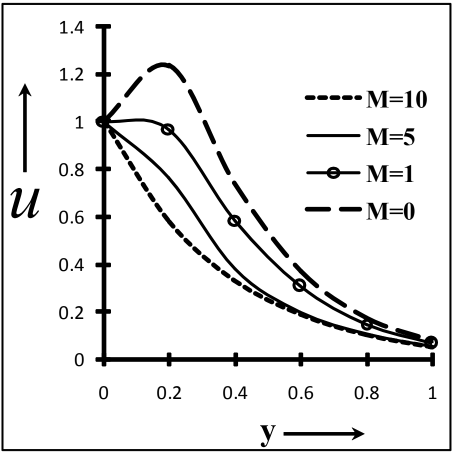


Fig. 10.4: Velocity field for the variations of Magnetic number against y

Results and discussion

In order to get a clear insight on the physics of the problem, a parametric study for magnetohydrodynamic force (M), radiation

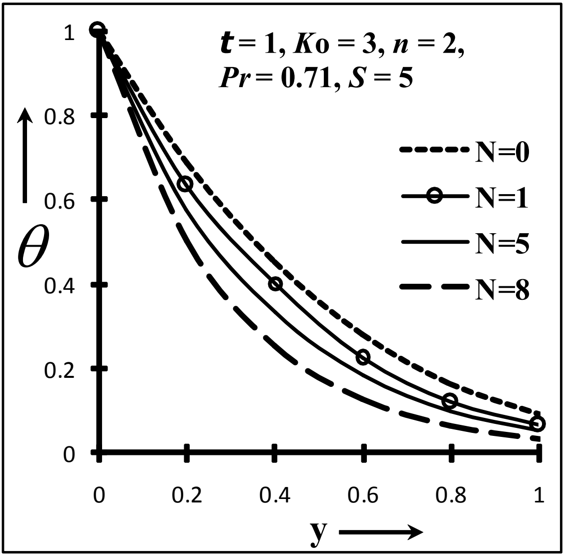


Fig. 10.5 Variations of radiation on Temperature field against *y*

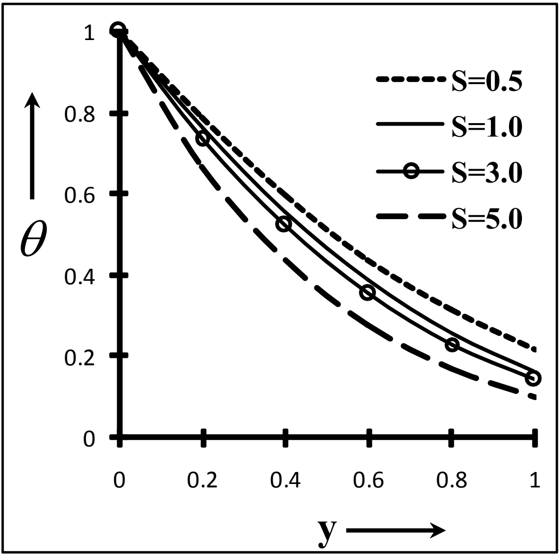


Fig. 10.6 Variations of heat source on Temperature fieldagainst *y*

parameter (N), Porosity parameter (K), Grashoff number (Gr), heat source (S) on the flow patterns is performed and the obtained numerical results are displayed with the help of graphical illustrations. During the numerical calculations the physical parameters are considered as Pr = 0.71 (diffusing air), $Gr = 5$ (thermal buoyancy forces are dominant over the viscous hydrodynamic forces in the boundary layer), $n = 5$, $S = 5$ and $\varepsilon = 0.005$. In Fig. 10.1 velocity profiles are displayed with the variations in porosity parameter K. From this figure, it is noticed that the velocity of the fluid accelerated from surface to free stream with the increase in the values of the porosity parameter K. physically, an increase in the permeability of porous medium leads the rise in the flow of fluid through it. When the holes of the porous medium become large, the resistance of the medium may be neglected. So that velocity at the insulated surface is observed to be zero and gradually it increases as it reaches the free stream. Figure 10.2 shows the effect of radiation N on the fluid velocity and it shows that the flow velocity accelerated with increasing values of radiation which is observed to be similar in case of porous permeability increases. The effect of velocity for different values of Grashoff Number is also presented in Fig. 10.3 it shows that the velocity decreases with increasing the Grashoff Number. The velocity distribution attains a distinctive maximum value in the vicinity of the plate surface and then decreases properly to approach the free stream value. As expected, the fluid velocity increases and the peak value becomes more distinctive due to increases in the thermal buoyancy effects represented by Gr. Figure 10.4 depicts the effect of the Hartmann number M on the fluid velocity and the results show that the presence of the magnetic force causes retardation of the fluid motion represented by general decreases in the fluid velocity. It is due the fact that magnetic force which is applied in the normal direction to the flow produces a drag force which is known as *Lorentz force*. The temperature profiles are calculated for different values of thermal radiation parameter ($N = 0, 1, 5, 8$) at time $t = 1$ and shown in Fig. 10.5. It is observed that the temperature increases with decreasing radiation parameter. Figure 10.6 reveals temperature variations with heat source S at $t = 1$. The temperature is observed to decrease with an increase in S.

Conclusion

Analytical solutions have been obtained to study the influence of thermal radiation, porosity of the medium and Heat source on transient convective heat transfer in boundary layer flow over a vertical porous plate immersed in a porous medium with time dependent suction velocity. A flux model has been employed to simulate thermal radiation effects by using Rosseland approximation. The analysis has shown that increasing thermal radiation/porosity/Grashoff number effects accelerate the flow velocity. Increasing heat source/radiation reduces the fluid temperature. The present study has been confined to Newtonian flow.

References

[1] Nield, D. A. and Bejan, A. Convection in Porous Media, Springer, New York, 1999.

[2] Ingham, D. B. and Pop, I (eds).Transport Phenomena in Porous Media, Pergamon, Oxford, 1998.

[3] Raptis, A., and Perdikis, C., Thermal radiation of an optically thin gray gas, *Int. J. Applied Mechanical Engg.*, **8**, 131–134 (2003).

[4] Jaiswal, B. S. and Soundalgekar, V. M. Oscillating plate temperature effects on a flow past an infinite vertical porous late with constant suction and embedded in a porous medium, *Heat and Mass Transfer*, **37**, 125-131 (2001).

[5] Bakier, A. Y. Thermal radiation effect of mixed convection from vertical surfaces in saturated porous media, *International Communications of Heat and Mass Transfer*, **28**, 119-126 (2001).

[6] Ahmed, S., Transient three dimensional flow through a porous medium with transverse permeability oscillating with time, *Emirate Journal for Engineering research*, **13** (3), 11-17 (2008).

[7] Ahmed, Sahin. Free Convective Transient Three-dimensional Flow through a porous medium oscillating with time in presence of Periodic Suction Velocity.*International Journal of Applied Mathematics and Mechanics*, **6** (11), 1-16 (2010).

[8] Ahmed, Sahin and Chamkha, A. J., Effects of chemical reaction, heat and mass transfer and radiation on the MHD flow along a vertical porous wall in the presence of induced magnetic field, *International Journal of Industrial Mathematics*, **2**(4), 245-261 (2010).

[9] Ahmed, Sahin and Liu, I-Chung, Mixed convective three-dimensional heat and mass transfer flow with transversely periodic suction velocity, *Int. J. Applied Mathematics and Mechanics*, **6**, 58-73 (2010).

[10] Makinde, O. D., Free convection flow with thermal radiation and mass transfer past a moving vertical porous plate, *International Communications in Heat and Mass Transfer*, **32**, 1411-1419 (2005).

Appendix

$$m_1 = \frac{P_r + \sqrt{P_r^2 + 4P_r S\left(1+\frac{4N}{3}\right)}}{2\left(1+\frac{4N}{3}\right)},\quad m_2 = \frac{P_r + \sqrt{P_r^2 + 4P_r S_1\left(1+\frac{4N}{3}\right)}}{2\left(1+\frac{4N}{3}\right)},$$

$$m_3 = \frac{1+\sqrt{1+4M_1}}{2},\quad m_4 = \frac{1+\sqrt{1+4M_2}}{2},\quad A_1 = \frac{-m_1 P_r}{\left(1+\frac{4N}{3}\right)m_1^2 - m_1 P_r + S_1 P_r},$$

$$A_2 = \frac{G_r}{m_1^2 - m_1 - M_1},\quad A_3 = \left(M_2 - \frac{1}{K_0}\right)(1+A_2),$$

$$A_4 = -(1+A_1)G_r,\quad A_5 = A_1 G_r + \frac{A_2}{k_0} - A_2 m_1$$

$$A_6 = \frac{A_3}{m_3^2 - m_3 - M_2},\quad A_7 = \frac{A_4}{m_2^2 - m_2 - M_2},\quad A_8 = \frac{A_5}{m_1^2 - m_1 - M_2}.$$

11

Evaluation of Leadership Qualities of the Heads of Schools and Students' Mathematics Achievements in Secondary Schools

Ashim Bora

Abstract

This paper describes the relationship between leadership quality of heads of schools and students' mathematics achievements in secondary schools. The secondary schools situated in Karbi Anglong district of Assam state of Indian Republic constitute the population of the study. A total of 48 secondary school situated at different educational blocks of the district are chosen as sample for collection of data for the study. A wide variant of school environments and academic achievements in mathematics were revealed in the study. A total of 80 mathematics teachers working in the selected schools participated in the study. All the teachers have at least two academic years of experience with the same head of the school. A research instrument is designed by the investigator to collect data related to leadership quality of the heads of the institutions. The data related to achievements in mathematics were collected from school records of HSLC examinations, annual examinations from class IX to X and test examinations in class X. Collected data are analysed with statistical tools. The study reveals that there exist a strong relationship between school heads leadership qualities and students' academic performances in the subject mathematics in secondary level. Heads' attitude towards mathematics directly affects the achievements of students in mathematics.

Keywords: *Leadership Qualities; Heads of Schools; Mathematics Achievements; Karbi Anglong.*

Introduction

School administration is a vital component of our education system. Each secondary school is headed by a Headmaster or Principal. The Head of a school is completely responsible foe planning and management of the school. The ways of use of available resources and the steps to mobilise the school resources are to be decided by the head of the school. Researchers in different countries are doing many works to find out the relationship between the leadership quality of the heads of educational institutions and the academic achievements of the institutions in the subject mathematics. Leithwood. K and Jantzi. D (2000) found that the quality of school heads are significantly related related to the academic achievements of the students. One of the major challenges tackled by the administrator of a school is to involve the teachers, parents and other related association towards the academic achievements for younger generation of the society. Darling Hammond L and Mac Laughlin M (1995) suggested that the head of institutions must build learning communities within their schools and engage the entire school community for achieving a compelled vision for their schools. In 2005 the works of Marzano, Waters and Mc Nulty demonstrated that there exists a highly positive relationship between behaviours of school heads and students achievement. The heads of institutions face different types of issues everyday. For good environment of school for teaching and learning and better performances of students, the head of schools must cultivate some innovative proposals and teaching methodology. Positive school climate is an important factor for higher academic performance of schools. According to Akin (1993), to be a successful administrator he or she must first understand schools climate and must know how to change a negative school climate to positive school climate. Fuller (1991) stated that " The role of the principal has become dramatically more complex, over loaded and unclear over the past decade". Academic achievement of the school students require improved instructional leadership. According to Hoy and Miskel (2001), " The principal of a healthy school provides dynamic leadership that is both task oriented and relations oriented. Such behaviour is supportive of teachers and yet provides directions and maintains high standards of performance". In the 21st century educational system principals are not only institutional leaders or master teachers , but also becoming transformational leaders. According to Ubben and Hughes " although the principal must address certain managerial tasks to ensure an efficient school, the task of the principal must be kept focused on activities which pave the way for higher student achievement". According to Kroze, the activities of a schoolhead should be centred on students' academic achievements. Fullan (1991) found in his research work "schools operated by Principals who were perceived by the teachers to be strong instructional leaders exhibited signficantly greater gain scores in achievements in reading and mathematics than did schools operated by average and weak instructional leader".

Objective of the Stdy

There are two leading objectives of the study. Firstly, to examine the leadership qualities of the heads of secondry schools of Karbi Anlong district of Assam. Secondly, try to findout the relationship between the school heads leadership qualities and students' academic achievements in matematics.

Research Quations

In this present study the researcher is trying to find the answers of the follwing queations.

Q1. What are the perceptions of teachers and the leadership Qualities of their school heads.

Q2. Is there exist any relationship between Leadership Qualities of Heads of schools and academic achievements of the students in mathematics.

Research Methodology:

The secondary and higher secondary schools situated in Karbi Anglong district of Assam constitute the population of the present study. Due to difficult geographical and time constraints it is not possible to visit each and every school of the district. The researcher has taken a sample of 48 schools situated at different educational blocks of the district for collection of data for the study. A total of 80 mathematics teachers working in the selected schools participated in the study. All the teachers have at least two academic years of experience with the same head of the school.

Table 11.1 Profile of schools

Urban	Rural	Urban	Rural
12	12	12	12

Table 11.2 Profile of respondents (Mathematics Teachers)

Gender		Age			Experience		
Male	**Female**	**< 35**	**35-45**	**> 45**	**< 5**	**5-15**	**> 15**
71	9	17	38	25	21	44	15
88.15%	11.25%	47.5%	31.25%	26.25%	55%	18.75%	

Research Instrument

A research instrument was designed by the investigator to collect the data related to Leadership qualities of the heads of the institutions. The mathematics teachers were requested to complete the research instrument. The questionnaire has eight dimensions and each dimension contains four items. Communication Skills, Comfort, Empathy, Decision Making, Influence, Self Management, Time Management and Commitment were the eight dimensions of the research instrument developed. Five point Likert Scale method was used in the research instrument. The five options are Strongly Agree (SA), Agree (A), Neutral (N), Disagree (D) and Strongly Disagree (SD). Weights assigned to each of the five levels are shown in the following table 11.4.

Table 11.4

Level of Response	Scores	
	Positive items	Negative Items
Strongly Agree(SA)	5	1
Agree(A)	4	2
Neutral(N)	3	3
Disagree(D)	2	4
Strongly Disagree(SD)	1	5

The pilot survey was done for reliability test of the research instrument on ten different schools situated at Diphu Town, which is the head quarter of Karbi Anglong. Twenty Five mathematics teachers working at these schools took part in the reliability test. Cronbach's Alpha reliability score for the research instrument was 0.72 which was acceptable. The collected data for the study were tabulated and analyzed with statistical tools, like mean, standard deviation, t-test. Data related to achievement in mathematics were collected from the offices of the participating schools. Mathematics marks obtained by the students in HSLC Exams, annual exams from class IX to X and test exams in Class X were used for measuring students' achievements in mathematics.

Data Analysis and Interpretation

Mean($\overline{x}$), standard deviation(σ) and t-test were applied for the interpretation of the collected data.

Table 11.4

Sl No	Statistical Measure	Responses				
		Urban	Rural	Private	Public	Total
1	Mean	**46.5**	29.4	**47.75**	23.25	40.25
2	S.D.	**16.08**	13	**13.49**	9.87	14.63

Academic Achievements in mathematics are shown in the table 11.5.

Table 11.5

Sl No	Statistical Measure	**Urban**	Rural	**Private**	Public	Total
1	Passed %	**61.45**	23.38	**68.23**	16.61	42.41
2	Mean	**17.5**	7.05	**13.15**	7.27	12.28
3	S.D.	**3.76**	4.50	**6.36**	2.47	6.82

The table 11.4 shows the perceptions of teachers on the Leadership Qualities of their School Heads The mean score is 40.25, which indicate that the perceptions of high school teachers in Karbi Anglong district of Assam possess only 44.72% positive attitude towards the leadership qualities of their respective school heads which is not so high. Teachers' perception responses are higher in urban areas ($\overline{x}$= 46.5, σ=16.08) than in rural areas(= 29.4, =13). Moreover, teachers of private sector schools have greater positive responses (= 47.75, =13.49) towards their heads' leadership qualities comparing to the teachers serving in public (Govt/ Provincialized) high schools(= 23.25, =9.87).

Table 11.5 shows Students' Academic Achievements in mathematics. Out of 1240 students studying in 45 secondary schools only 526 passed in the subject in HSLC examination of 2011 examination. The pass percentage is 42.41, mean($\bar{x}$) is 12.28, S.D. (σ) is 6.82. Out of 620 Urban students (No of Schools=22), 381 passed in mathematics (= 17.5, =3.76, %=61.45). In rural area from equal number of Schools only 145 students could cleared their mathematics subject(= 7.05, =4.50, %=23.38).There is a vast difference in pass percentage. On the other hand the students studying in Private sector schools do well in the examination. Out of 620 students selecting from 23 private schools, a total of 423 passed in the subject mathematics (= 13.15, =6.36, %=68.23). Compare to that only 103 students passed from 22 schools situated in rural areas of Karbi Anglong(= 7.27, =2.47, %=16.61).

Discussion and Conclusion

In this study the researcher sought to investigate the relationship between Leadership Qualities of heads of schools of Karbi Anglong and academic achievements of students in mathematics. Results from the study show that mean score of heads' leadership qualities is only 40.25 out of maximum score 90 , which indicates that heads' have less leadership qualities. Further study may be carried out to find out the reasons of heads' less positive leadership qualities of this ST dominated Region of India.

The present study reveals that there exist a strong relationship between leadership qualities and students' academic achievements in mathematics.

***References*:**

1. Ahmed, S and Bora, A., *Gender Differences and Achievements in Mathematics among the Students in High School Examination in Diphu Town of Karbi Anglong District of Assam, India. International Journal of Mathematical Education,* **1**(1), 57-62 (2011).
2. Ahmed, S., and Bora, A., The Relationship Between Teachers' Attitude About Teaching Mathematics And Students' Mathematics Achievement In India, This paper has been accepted for publication in the *Journal of International Education Research, Colorado*, 80128, USA (2014).
3. Akin, J., *The effects of site culture on reform*, High School Magazine, **1**(1), 29 (1993).
4. Darling-Hammond, Linda, and Milbrey W. McLaughlin., Policies that

Support Professional Development in an Era of Reform, *Phi Delta Kappan*, **76** (8), 597–604 (1995).

5. Fuller, B., Growing Up Modern: The Western State Builds Third World Schools, New York: Routledge, (1991).

6. Hoy, W. K., and Miskel, C. G., Educational Administration: Theory, Research and Practice, 6th edition, New York, McGraw Hill (2001).

7. Kroeze, D., District context and its impact on the instructional leadership role of the elementary school principal, Unpublished doctoral dissertation, University of Chicago, Chicago, IL. ERIC Document No. T-31945 (1992).

8. Leithwood, K., and Jantzi, D., Principal and teacher leader effects: a replication, School Leadership and Management, **20**(4), 415–434 (2000).

9. Marzano, R. J., Waters, T., and McNulty, B.A., School leadership that works: From research to results. Alexandria, Association for Supervision and Curriculum Development and Denver,CO: Mid-continent Research for Education and Learning, (2005).

12

Fibonacci sequence and morphology of plants

Abhijit Konch

Abstract

This introductory paper wants to explore how Fibonacci sequence appears in the plants morphology (especially in some domestic plants). An attempt is also made to exhibit how widespread rich mathematical ideas are in the natural world.

Key words: *Fibonacci sequence, Plants Morphology, Phyllotaxis etc.*

Introduction

Fibonacci sequence and Numbers: This is the sequence where every number in the sequence (after the second) is the sum of the previous 2 numbers. The series of numbers is 1,1,2,3,5,8,13,21,34,……. These numbers in the sequence are called Fibonacci numbers and the sequence is known as Fibonacci sequence.

Golden Ratio Two quantities are in the **golden ratio** if the ratio of the sum of the quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one. Usually the Greek letter *Phi(ö)* is used to denote *Golden Ratio*.

Golden ratio =1.61803399 (approx.)

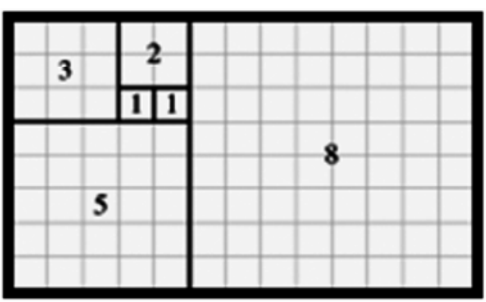
i.e. if a and b are two numbers such that a is greater than b, then a and b are said to be in the golden ratio if a+b is to a as a is to b.

Fig. 12.1

Relation between Fibonacci number and Golden Ratio

The ratio of successive pair of Fibonacci Number is the golden ratio. If we divide consecutive Fibonacci Numbers the results converge on the Golden Ratio. The larger the Fibonacci Number the closer the results are to the Golden Ratio.

Fibonacci Rectangle: If we start with two small squares of size 1 next to each other. On top of both of these, draw a square of size 2 (=1+1). We can now draw a new square - touching both a unit square and the latest square of side 2 - so having sides 3 units long; and then another touching both the 2-square and the 3-square (which has sides of 5 units).



Source:http://en.wikipedia.org/wiki/Fibonacci_number

Fig. 12.2

We can continue adding squares around the picture; **each new square having a side which is as long as the sum of the latest two square's sides**. This set of rectangles whose sides are two successive Fibonacci numbers in length and which are composed of squares with sides which are Fibonacci numbers, are called **Fibonacci Rectangles**.

Fibonacci Spiral or Golden spiral: Here is a spiral drawn in the squares, a quarter of a circle in each square. It is one kind of spiral that appears often in nature. Such spirals are seen in the shape of shells of snails and sea shells and in the arrangement of seeds on flowering plants too.

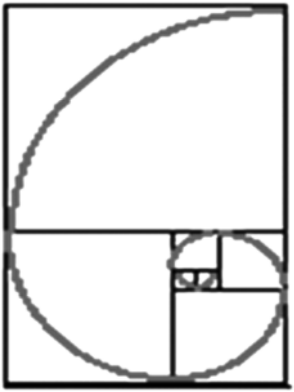


Image Source: http://mathworld.wolfram.com/GoldenRatio.html

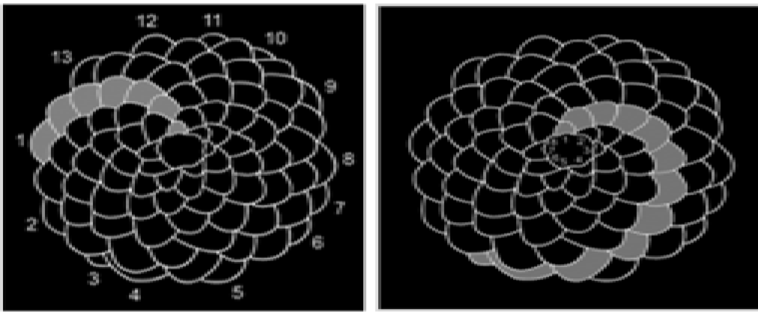
Fig. 12.3

In mathematics, a spiral is a curve which emanates from a central point, getting progressively farther away as it revolves around the point. Fibonacci Spiral is a **logarithmic spiral**, **equiangular spiral** or **growth spiral.** In polar coordinates (r,θ) the logarithmic curve can be written as $r = ae^{b\theta}$ where a and b are positive real numbers.

Phyllotaxy: Phyllotaxy, a subdivision of plant morphology, is the study of the arrangement of repeated units such as leaves around a stem, scales on a pine cone or on a pine apple, florets in the head of a daisy and seeds in a sunflower. The Fibonacci sequence is also observed and studied in a different branch in phyllotaxy. The Golden number Ö =("5+1)/2, is also important in phyllotaxis study. The observation of the Fibonacci sequence in botany constituted a mystery which served as main spur to the development of the subject. We can express this mystery by saying that the number of spirals in observed systems of opposed families of spirals (as seen in daisies and sunflowers for example) are generally consecutive terms of the Fibonacci sequence.

Fibonacci numbers in plant spirals: Sunflower, pine apples, Pine cones, what do they have in common other than the fact that they are all plants is that each one is formed of opposite sets of spirals. A Pine apple has 8 spirals going in the opposite direction.

A pine cone – 8 and 13. A sun flower -55 and 89. Interestingly, 8, 13, 21 and 34 all are consecutive terms of Fibonacci sequence.

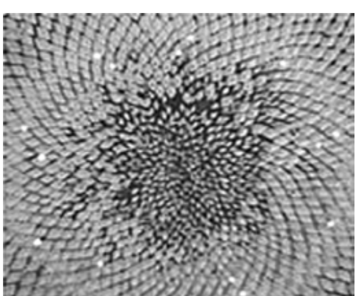


Source:http://britton.disted.camosun.bc.ca/fibslide/jbfibslide.htm

Fig. 12.4

In this pine-cone, 8-spirals are in the clockwise direction and 13-spirals are in the anti-clockwise direction.

Fibonacci in the Sunflower: The Sunflower, shown below, is a perfect example of the Fibonacci sequence and the corresponding 'Golden ratio' appearing in nature. Firstly see how the florets are arranged in a spiral pattern both in a clockwise and counterclockwise fashion. There are 89 spirals that turn clockwise and 55 spirals that turn counterclockwise. The counterclockwise spirals appear to grow according to the Golden ratio.



Source:www.goldennumber.net

Fig. 12.5

Fibonacci numbers in flower petals: Fibonacci numbers occurs even in the arrangement of flowers-

Source: http://jwilson.coe.uga.edu/emat6680/parveen/fib_nature.htm

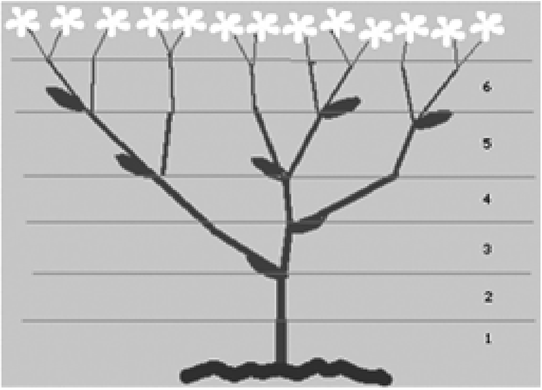
Fig. 12.6

3 petals	*Lilies*
5 petals	*Buttercups, Roses*
8 petals	*Delphinium*
13 petals	*Marigolds*
21 petals	*Black-eyed susans*
34 petals	*Pyrethrum*
55/89 petals	*Daisies*

Fibonacci numbers and leaves around a stem: The Fibonacci series also shows up in the distribution of leaves around a stem. If we count the number of leaves needed to complete a spiral around a stem before they align with each other, the resulting number is likely from the Fibonacci Series. This phenomenon of phyllotaxis is a result of a growth pattern that insures that each leaf gets optimal access to sunlight. Some other works suggests that these structures minimize energy in the creation and growth of the organism, which is essential for their long term survival as a species.

A cactus has no leaves: A cactus has no leaves, but we can see the organization of buds on its stem. Interestingly, it is a spiral arrangement. This spiral is based on the Fibonacci series.

Fibonacci numbers in plant branching:



Source:http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html

Fig. 12.7

Fibonacci numbers in plant sections:

Banana have 3 and Apple have 5.

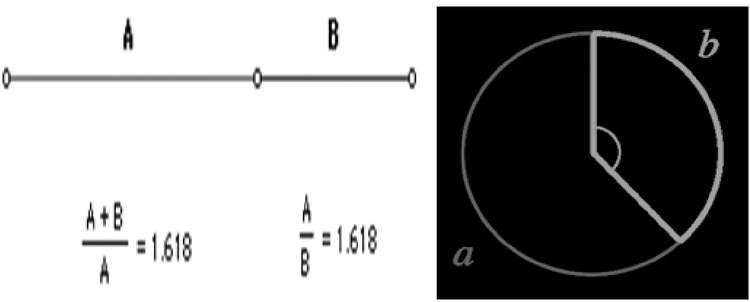
Source:http://jwilson.coe.uga.edu/emat6680/parveen/fib_nature.htm

Fig. 12.8

Golden angle and the spiral leaf arrangement found in many plants: In spiral phyllotaxis, the number of visible spirals is most often two successive elements of the Fibonacci sequence. When this occurs, the angle between successive leaves or botanical element is close to the Golden Angle - about 137.50, related to the golden mean.

If we take a line that is split in Golden Proportion and grabbed the ends of it with our hands and bent in around into a circle we would get a circle whose circumference is split to the golden Ratio. From this we can find the “Golden Angle”, which is 137.5 degrees**.**

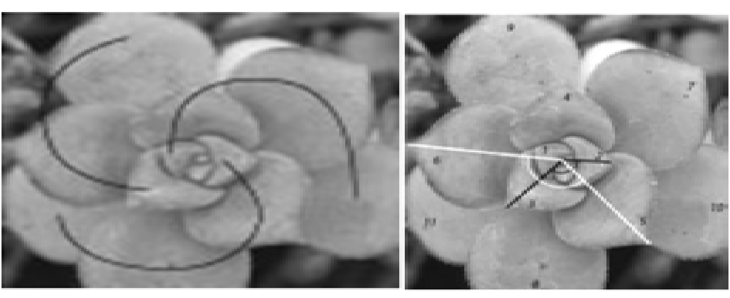
Golden Angle=360×(2- Ö)H” 137.5…



Source: http://en.wikipedia.org/wiki/Golden_angle

Fig. 12.9

We find that many plants, trees and flowers (not all) tend to branch to the Golden Angle. In other words, for every 137.5 degree turn a new leaf or branch forms.



Source:www.goldennumber.net

Fig. 12.10

Fibonacci numbers in vegetables: In Cauliflower as for example, each floret is peaked and is an identical but smaller version of the whole thing and this makes the spirals easy to see.

Conclusion

Immanul Kant once wrote" I assert only that in every particular Nature-study, only so much real science can be encountered as there is mathematics to be found in it." Mathematics has now entered into the study of phyllotaxy in variety of ways. Many branches of mathematics like statistics, calculus, differential equation etc. have been used. Familiarity with Mathematics has become a must for botanists, interested in plant morphogenesis.

References

1. Barnsley, Michael, Fractals Everywhere Morgan Kaufmann, (2000).
2. Kaplan, Donald R., The Science of Plant Morphology: Definition, History and Role in Modern Biology,American Journal of Botany 88(10), (2001).
3. Sheldrake, Rupert, The presence of the past: Morphic Resonance and the Habits of Nature, Park Street Press (1995).
4. www. tcpd.org
5. www. Tcpdpodcast.org

13

Generalized Class of Transformed Ratio-Product Estimation of Finite Population Mean in Sample Surveys

W.Warseen Chanu
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Abstract

We have considered a generalized class of ratio-product estimators for estimating a finite population mean using the transformation on a and b. The asymptotically optimum estimator in this case is identified, along with its approximate mean square error. Further with the suitable choices of the constants a and b., we have some members of the proposed class of estimators. It has been shown that the proposed class of estimators is more efficient than the other members of the estimator. An empirical study is also carried out to demonstrate the merits of the proposed estimator over other estimators.

Keywords: *Ratio-product, Asymptotically optimum estimator, Mean square error, Estimators.*

Introduction

In sample surveys, auxiliary information is used at both selection as well as estimation stages to improve the efficiency of the estimators. When the correlation between study variate and the auxiliary variate is positive, the ratio method of estimation is used for estimating the population mean. On the other hand, if the correlation is negative, the product method of estimation envisaged by [5] is used. It has been theoretically established that, in general, the linear regression estimator is more efficient than the ratio (product) estimator except when the regression line of study variable y on auxiliary variable x passes through the neighborhood of the origin, in which case the efficiencies of these estimators are almost equal. [10] and [3] used coefficient of variation along with the population mean of auxiliary variate. Use of coefficient of Kurtosis of auxiliary variate has also been in practice for improving the efficiency of the estimators of finite population mean. [12] and [8] utilized coefficient of kurtosis of auxiliary variate for estimating the finite population mean. [4] and [7] motivated authors to propose a ratio-cum-product of finite population mean.

Let $U = (U_1, U_2, ..., U_N)$ be the finite population of size N and y and x be the study and auxiliary variates respectively. A sample of size n is drawn using simple random sampling without replacement to estimate the population mean $\bar{Y} = \sum_{i=1}^{N} y_i / N$ of study variate y.

The classical ratio and product estimators for $\bar{Y}$ are respectively given by

$$\bar{y}_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \quad \text{and} \quad \bar{y}_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)$$

where $\bar{x} = \sum_{i=1}^{n} x_i / n$ and $\bar{y} = \sum_{i=1}^{n} y_i / n$. Here it is assumed that $\bar{X} = \sum_{i=1}^{N} x_i / N$, population mean of auxiliary variate is known.

The usual regression estimator is given by

$$t_{RG} = \bar{y} + \hat{\beta}\left(\bar{X} - \bar{x}\right)$$

where $\hat{\beta}$ is the sample estimate of population regression coefficient β of y on x.

[7] suggested ratio-product estimation of a finite population mean as

$$t_{RP} = \bar{y}\left\{k\frac{\bar{X}}{\bar{x}} + (1-k)\frac{\bar{x}}{\bar{X}}\right\}$$ where k is a real constant.

[10] suggested a ratio estimator of population mean using coefficient of variation C_x of auxiliary variate as

$$\hat{\bar{Y}}_{SD} = \bar{y}\left(\frac{\bar{X} + C_x}{\bar{x} + C_x}\right)$$

[6] used known value S_X, standard deviation of x in forming a modified product estimator as

$$\hat{\bar{Y}}_{GS} = \bar{y}\left(\frac{\bar{x} + S_x}{\bar{X} + S_x}\right)$$

[9] assumed advanced knowledge of the coefficient of correlation ρ to form a modified ratio estimator of the population mean of y as

$$\hat{\bar{Y}} = \bar{y}\left(\frac{\bar{X} + \rho}{\bar{x} + \rho}\right).$$

Utilizing the information on coefficient of variation C_x and coefficient of Kurtosis $\beta_2(x)$ of the auxiliary variate x, [12] proposed the following ratio and product estimators respectively

$$\hat{\bar{Y}}_{UP1} = \bar{y}\left(\frac{\bar{X}C_x + \beta_2(x)}{\bar{x}C_x + \beta_2(x)}\right), \quad \hat{\bar{Y}}_{UP2} = \bar{y}\left(\frac{\bar{x}C_x + \beta_2(x)}{\bar{X}C_x + \beta_2(x)}\right)$$

[8] defined a ratio estimator using coefficient of kurtosis $\beta_2(x)$ which is

$$\hat{\bar{Y}}_{S} = \bar{y}\left(\frac{\bar{X} + \beta_2(x)}{\bar{x} + \beta_2(x)}\right)$$

[11] defined a modified ratio-cum-product using known coefficient of variation and coefficient of Kurtosis as

$$\hat{\bar{Y}}_M = \bar{y}\left[\alpha\left(\frac{\bar{X}C_x + \beta_2(x)}{\bar{x}C_x + \beta_2(x)}\right) + (1-\alpha)\left(\frac{\bar{X}C_x + \beta_2(x)}{\bar{x}C_x + \beta_2(x)}\right)\right]$$ where α is suitably chosen scalar.

Motivated by [7] and [11] we have proposed a generalized class of ratio-cum-product estimators for estimating the population mean $\bar{Y}$.

The Proposed Estimator

The proposed generalized class of ratio-product estimators of population mean $\bar{Y}$ is

$$\hat{\bar{Y}}_{SW} = \bar{y}\left[\alpha\left(\frac{a\bar{X} + b}{a\bar{x} + b}\right) + (1-\alpha)\left(\frac{a\bar{x} + b}{a\bar{X} + b}\right)\right] \tag{1}$$

where α is a suitably chosen scalar to be determined such that the mean-square error (MSE) of $\hat{\bar{Y}}_{SW}$ is a minimum.

To obtain the bias and MSE of $\hat{\bar{Y}}_{SW}$ to the first degree of approximation, we write:

$\bar{y} = \bar{Y}\left(1 + e_0\right)$ and $\bar{x} = \bar{X}\left(1 + e_1\right)$ such that

$E(e_0) = E(e_1) = 0$ and under an SRSWOR, we have

$$\left.\begin{aligned} E(e_0^2) &= \left(\frac{1-f}{n}\right)C_Y^2, E(e_1^2) = \left(\frac{1-f}{n}\right)C_X^2 \\ E(e_0 e_1) &= \left(\frac{1-f}{n}\right)\rho_{YX} C_Y C_X \end{aligned}\right\} \tag{2}$$

where $f = \dfrac{n}{N}$ is the sampling fraction,

$$C_Y = S_Y / \bar{Y}, \quad C_X = S_X / \bar{X}, \quad \rho_{XY} = S_{XY} / S_X S_Y, \quad K_{YX} = \rho_{XY} C_Y / C_X$$

$$S_X^2 = \sum_{i=1}^{N} (x_i - \bar{X})^2 / (N-1), \quad S_Y^2 = \sum_{i=1}^{N} (y_i - \bar{Y})^2 / (N-1), \quad S_{XY} = \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{X}) / (N-1)$$

Expressing (1) in terms of e_i ‘s, $i = 1, 2$, we have

$$\hat{\bar{Y}}_{SW} = \bar{Y}(1+e_0)\left[\alpha(1+\phi e_1)^{-1} + (1-\alpha)(1+\phi e_1)\right]$$

Assuming $|\phi e_1| < 1$ so that we may expand $(1+\phi e_1)^{-1}$ as a series in power of e_1. Expanding, multiplying out and retaining terms of e ’s to the second degree, we obtain

$$\hat{\bar{Y}}_{SW} = \bar{Y}\left[1 + e_0 + \phi e_1 + \phi e_0 e_1 + \alpha\left(-2\phi e_1 + \phi^2 e_1^2 - 2\phi e_0 e_1\right)\right]$$

$$\hat{\bar{Y}}_{SW} - \bar{Y} = \bar{Y}\left[e_0 + \phi e_1 + \phi e_0 e_1 + \alpha\left(-2\phi e_1 + \phi^2 e_1^2 - 2\phi e_0 e_1\right)\right] \tag{3}$$

where $\phi = \dfrac{a\bar{X}}{a\bar{X} + b}$

Taking expectations on both the sides of eq (3) and using the results from eq (2), we obtain the bias of $\hat{\bar{Y}}_{SW}$ to the first degree of approximation as follows

$$B\left(\hat{\bar{Y}}_{SW}\right) = \bar{Y}\left(\frac{1-f}{n}\right)\phi C_X^2\left[K_{YX} + \alpha\left(\phi - 2K_{YX}\right)\right] \tag{4}$$

Thus, the estimator $\hat{\bar{Y}}_{SW}$ with $\alpha = K_{YX} / \left(2K_{YX} - \phi\right)$ is almost unbiased. It is observed from (4) that the bias of $\hat{\bar{Y}}_{SW}$ is negligible for large sample.

Squaring both the sides of eq (3) and retaining terms to the second degree, we have

$$\left(\hat{\bar{Y}}_{SW} - \bar{Y}\right)^2 = \bar{Y}^2\left[e_0^2 + \phi^2(1-2\alpha)^2 e_1^2 + 2\phi(1-2\alpha)e_0 e_1\right] \tag{5}$$

Taking expectations on both the sides of eq (5) and using results from eq (2), we obtain the MSE of $\hat{\bar{Y}}_{SW}$ to the first degree of approximation as

$$MSE\left(\hat{\bar{Y}}_{SW}\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2+\phi(1-2\alpha)^2 C_X^2+2\phi(1-2\alpha)\rho_{YX}C_YC_X\right] \tag{6}$$

which is minimized when

$$\alpha=\frac{1}{2}\left(1+\frac{K_{YX}}{\phi}\right)=\alpha_{opt} \quad (say) \tag{7}$$

Substituting eq (7) in eq (1), we get the asymptotically optimum estimator(AOE) for $\hat{\bar{Y}}_{SW}$ as

$$\hat{\bar{Y}}_{SW}^{opt}=\frac{\bar{y}}{2}\left[\left(1+\frac{K_{YX}}{\phi}\right)\left(\frac{a\bar{X}+b}{a\bar{x}+b}\right)+\left(1-\frac{K_{YX}}{\phi}\right)\left(\frac{a\bar{x}+b}{a\bar{X}+b}\right)\right] \tag{8}$$

Putting the value of α in eq (4) and eq (6), we obtain the optimum bias and the MSE of $\hat{\bar{Y}}_{SW}^{opt}$ as follows

$$B\left(\hat{\bar{Y}}_{SW}^{opt}\right)=\bar{Y}\left(\frac{1-f}{n}\right)\phi C_X^2\left[\frac{K_{YX}}{2}-\frac{K_{YX}^2}{\phi}+\frac{\phi}{2}\right] \tag{9}$$

$$MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)C_Y^2\left(1-\rho_{YX}^2\right) \tag{10}$$

From equation eq (9), we note that the $B\left(\hat{\bar{Y}}_{SW}^{opt}\right)$ is zero if

$K_{YX}=\phi \Rightarrow \beta=\phi R$, where $R=\dfrac{\bar{Y}}{\bar{X}}$ and $\beta=\rho_{YX}\dfrac{S_Y}{S_X}$

It is clear that the mean square error of $\hat{\bar{Y}}_{SW}^{opt}$ is same as that of the variance of the ususl linear regression estimator $\bar{y}_{lr}=\bar{y}+\hat{\beta}\left(\bar{X}-\bar{x}\right)$, where $\hat{\beta}$ is the sample regression coefficient of y on x.

Some members of the proposed class of estimators$\left(\hat{\bar{Y}}_{SW}\right)$}

The following are the estimators of the population mean which can be obtained by suitable choices of α, a and b.

Estimator	***a***	***a***	***b***
$T_1 = \bar{y}\left(\frac{\bar{X}+C_X}{\bar{x}+C_X}\right)$	1	1	C_X
$T_2 = \bar{y}\left(\frac{\bar{x}+S_X}{\bar{X}+S_X}\right)$	0	1	S_X
$T_3 = \bar{y}\left(\frac{C_X\bar{X}+\beta_2(x)}{C_X\bar{x}+\beta_2(x)}\right)$	1	C_X	$\beta_2(x)$
$T_4 = \bar{y}\left(\frac{\bar{X}+\rho}{\bar{x}+\rho}\right)$	1	1	r
$T_5 = \bar{y}\left(\frac{\bar{x}+C_X}{\bar{X}+C_X}\right)$	0	1	C_X
$T_6 = \bar{y}\left(\frac{\bar{X}+S_X}{\bar{x}+S_X}\right)$	1	1	S_X
$T_7 = \bar{y}\left(\frac{\bar{x}C_X+\beta_2(x)}{\bar{X}C_X+\beta_2(x)}\right)$	0	C_X	$\beta_2(x)$
$T_8 = \bar{y}\left(\frac{\bar{x}+\rho}{\bar{X}+\rho}\right)$	0	1	r
$T_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)$	1	1	0
$T_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)$	0	1	0

where C_X is the coefficient of variation, S_X is the standard deviation, $\beta_2(x)$ is the coefficient of kurtosis of the auxiliary variable x and ρ is the correlation coefficient between the study variable y and the auxiliary variable x.

The expressions of bias and mean square error of the above estimators can be obtain by mere substituting the values of α, a and b in eq (4) and eq (6), respectively. Up to the first degree of approximation, the bias and MSE of the above mention estimators are given as

$$B(T_1) = \bar{Y}\left(\frac{1-f}{n}\right)\left[\lambda_1^2 C_X^2 - \lambda_1 \rho_{YX} C_Y C_X\right] \tag{11}$$

$$MSE(T_1) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2 + C_X^2 \lambda_1^2 - 2\lambda_1 \rho_{YX} C_Y C_X\right] \tag{12}$$

$$B(T_2) = \bar{Y}\left(\frac{1-f}{n}\right)\lambda_2 \rho_{YX} C_Y C_X \tag{13}$$

$$MSE(T_2) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2 + \lambda_2^2 C_X^2 + 2\lambda_2 \rho_{YX} C_Y C_X\right] \tag{14}$$

$$B(T_3) = \bar{Y}\left(\frac{1-f}{n}\right)\left[\lambda_3^2 C_X^2 - \lambda_3 \rho_{YX} C_Y C_X\right] \tag{15}$$

$$MSE(T_3) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2 + \lambda_3^2 C_X^2 - 2\lambda_3 \rho_{YX} C_Y C_X\right] \tag{16}$$

$$B(T_4) = \bar{Y}\left(\frac{1-f}{n}\right)\left[\lambda_4^2 C_X^2 - \lambda_4 \rho_{YX} C_Y C_X\right] \tag{17}$$

$$MSE(T_4) = \bar{Y}\left(\frac{1-f}{n}\right)\left[C_Y^2 + \lambda_4^2 C_X^2 - 2\lambda_4 \rho_{YX} C_Y C_X\right] \tag{18}$$

$$B(T_5) = \bar{Y}\left(\frac{1-f}{n}\right)\lambda_1 C_X^2 K_{YX} \tag{19}$$

$$MSE\left(T_5\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2+\lambda_1^2C_C^2+2\lambda_1\rho_{YX}C_YC_X\right] \tag{20}$$

$$B\left(T_6\right)=\bar{Y}\left(\frac{1-f}{n}\right)\left[\lambda_2^2C_X^2-\lambda_2\rho_{YX}C_YC_X\right] \tag{21}$$

$$MSE\left(T_6\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2+\lambda_2^2C_X^2-2\lambda_2\rho_{YX}C_yC_X\right] \tag{22}$$

$$B\left(T_7\right)=\bar{Y}\left(\frac{1-f}{n}\right)\left[\lambda_3\rho_{YX}C_YC_X\right] \tag{23}$$

$$MSE\left(T_7\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2+\lambda_3^2C_X^2+2\lambda_3\rho_{YX}C_YC_X\right] \tag{24}$$

$$B\left(T_8\right)=\bar{Y}\left(\frac{1-f}{n}\right)\lambda_4\rho_{YX}C_YC_X \tag{25}$$

$$MSE\left(T_8\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2+\lambda_4^2C_X^2+2\lambda_4\rho_{YX}C_YC_X\right] \tag{26}$$

$$B\left(T_9\right)=\bar{Y}\left(\frac{1-f}{n}\right)C_X^2\left(1-K_{YX}\right) \tag{27}$$

$$MSE\left(T_9\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2+C_X^2-2\rho_{YX}C_YC_X\right] \tag{28}$$

$$B\left(T_{10}\right)=\bar{Y}\left(\frac{1-f}{n}\right)\rho_{YX}C_YC_X \tag{29}$$

$$MSE\left(T_{10}\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)\left[C_Y^2+C_X^2+2\rho_{YX}C_YC_X\right] \tag{30}$$

\where $\lambda_1=\dfrac{\bar{X}}{\bar{X}+C_X}$, $\lambda_2=\dfrac{\bar{X}}{\bar{X}+S_X}$, $\lambda_3=\dfrac{C_X\bar{X}}{C_X\bar{X}+\beta_2(x)}$, $\lambda_4=\dfrac{\bar{X}}{\bar{X}+\rho}$

Under simple random sampling without replacement (SRSWR), variance of sample mean $\bar{y}$ estimator is given by

$$V(y)=\bar{Y}^2\left(\frac{1-f}{n}\right)C_Y^2 \tag{31}$$

Estimator based on estimated 'Optimum'.

The AOE $\hat{\bar{Y}}_{SW}^{opt}$ requires prior knowledge about K_{YX}. In practice it would be difficult to obtain the exact value of K_{YX} as it involves unknown parameters C_Y, C_X and ρ. In practical sample surveys, prior estimates of C_Y, C_X and ρ can be obtained with reasonable accuracy either from a pilot survey, or past data, or experience or even from expert guesses by specialists in the field concerned. However, it is somewhat doubtful that any of these procedures can provide a precise guide to the value of K_{YX}, since this depends on precisely specifying the value of several. Furthermore, estimated or specified values may not be exact owing to changes to frame, population size and so on. Then the only alterative left to the investigator is to replace K_{YX} in eq (8) by its consistent estimate K_{YX}^* computed from the data at hand. Thus the estimator based on the estimated optimum is

$$\left(\hat{\bar{Y}}_{SW}^{opt*}\right)=\frac{\bar{y}}{2}\left[\left(1+\frac{K_{YX}^*}{\phi}\right)\left(\frac{a\bar{X}+b}{a\bar{x}+b}\right)+\left(1-\frac{K_{YX}^*}{\phi}\right)\left(\frac{a\bar{x}+b}{a\bar{X}+b}\right)\right] \tag{32}$$

where $K_{YX}^*=\left(\frac{s_{xy}}{s_x^2}\right)\bar{X}/\bar{y}=\frac{\hat{\beta}}{\hat{R}}$ where $\bar{X}$ is known,

$$s_{xy}=\sum_{i=1}^{n}\frac{(x_i-\bar{x})(y_i-\bar{y})}{n-1},\quad s_x^2=\sum_{i=1}^{n}\frac{\left(x_i-\bar{x}^2\right)}{n-1}\ \text{and}\ \hat{R}=\bar{y}/\bar{X}.$$

To obtain the MSE of $\left(\hat{\bar{Y}}_{SW}^{opt*}\right)$, we write

$$K_{YX}^*=K_{YX}\left(1+e_2\right)\ \text{with}\ E\left(K_{YX}+o(n^{-1})\right)$$

Expressing $\left(\hat{\bar{Y}}_{SW}^{opt*}\right)$ in terms of e's , we have

$$\left(\hat{\bar{Y}}_{SW}^{opt*}\right)=\frac{\bar{Y}}{2}(1+e_0)\left[\left\{1+\frac{K_{YX}(1+e_2)}{\phi}\right\}(1+\phi e_1)^{-1}+\left\{1-\frac{K_{YX}(1+e_2)}{\phi}\right\}(1+\phi e_1)\right]$$

$$=\bar{Y}\left[1+e_0-K_{YX}e_1-K_{YX}e_1e_2+\frac{\phi}{2}(\phi+K_{YX})e_1^2e_0e_1\right]$$

where e_0 and e_1 are as defined in section 2. The MSE of $\left(\hat{\bar{Y}}_{SW}^{opt*}\right)$ is

$$MSE\left(\hat{\bar{Y}}_{SW}^{opt*}\right)=E\left(\hat{\bar{Y}}_{SW}^{opt*}-\bar{Y}\right)^2=\bar{Y}^2E\left[e_0-K_{YX}e_1\right]^2$$

$$=\bar{Y}^2\left[e_0^2+K_{YX}^2e_1^2-2K_{YX}e_0e_1\right]=\frac{1-f}{n}\bar{Y}^2\left[C_Y^2+K_{YX}^2C_X^2-2K_{YX}\rho_{YX}C_YC_X\right]$$

$$=\frac{1-f}{n}S_Y^2(1-\rho_{YX})^2$$

which is same as that of $\hat{\bar{Y}}_{SW}^{opt}$ i.e. MSE $\hat{\bar{Y}}_{SW}^{opt}$ =MSE $\hat{\bar{Y}}_{SW}^{opt*}$.

Thus, we have proved the result that the MSE of the estimator $\hat{\bar{Y}}_{SW}^{opt*}$ in eq (32) based on the estimated optimum, to the first degree of approximation, is the same as that of $\hat{\bar{Y}}_{SW}^{opt}$ in eq (10).

Efficiency Comparisons

Comparing (12) and (10) we see that

$$MSE(T_1)-MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)(\lambda_1C_X-\rho_{YX}C_YC_X)^2>0 \tag{33}$$

Comparing (14) and(10), we see that

$$MSE(T_2)-MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right)=\bar{Y}^2\left(\frac{1-f}{n}\right)(\lambda_2C_X+\rho_{YX}C_Y)^2>0 \tag{34}$$

Comparing (16) and (10), we see that

$$MSE(T_3) - MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left(\lambda_3 C_X - \rho_{YX} C_Y^2\right)^2 > 0 \tag{35}$$

Comparing (18) and (10) we see that

$$MSE(T_4) - MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left(\lambda_4 C_X - \rho_{YX} C_Y\right)^2 > 0 \tag{36}$$

Comparing (20) and (10), we see that

$$MSE(T_5) - MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left(\lambda_1 C_X + \rho_{YX} C_Y^2\right)^2 > 0 \tag{37}$$

Comparing (22) and (10), we see that

$$MSE(T_6) - MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left(\lambda_2 C_X - \rho_{YX} C_Y\right)^2 > 0 \tag{38}$$

Comparing (24) and (10), we see that

$$MSE(T_7) - MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left(\lambda_3 C_X + \rho_{YX} C_Y\right)^2 > 0 \tag{39}$$

Comparing (26) and (10), we see that

$$MSE(T_8) - MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left(\lambda_4 C_X + \rho_{YX} C_Y\right)^2 > 0 \tag{40}$$

Comparing (28) and (10), we see that

$$MSE(T_9) - MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left(C_X - \rho_{YX} C_Y\right)^2 > 0 \tag{41}$$

Comparing (30) and (10), we see that

$$MSE(T_{10}) - MSE\left(\hat{\bar{Y}}_{SW}^{opt}\right) = \bar{Y}^2\left(\frac{1-f}{n}\right)\left(C_X + \rho_{YX} C_Y\right)^2 > 0. \tag{42}$$

It is seen from the above eqs (33), (34), (35), (36), (37), (38), (39), (40), (41) and (42) that the estimator $\left(\hat{\bar{Y}}_{SW}^{opt}\right)$ is more efficient than the other estimators so reduced from the proposed estimator.

Empirical Study

To analyze the performance of the proposed estimator in comparison to the other estimators, four natural population data sets are being considered. The description of the data is given below:

Population-I Source:[2]

The population consists of 278 village town/wards under Gajole police station of Malada district of West Bengal, India, (in fact only those villages of town/wards have been considered which are shown as inhabited and common in both census 1961 and census 1971 list). The variates considered are:

x : The number of agricultural laborers for 1961.

y : The number of agricultural laborers for 1971.

$\bar{Y}$ =39.0680, $\bar{X}$ =25.1110, C_X =1.6198, C_Y =1.4451, ρ =0.7213, $\beta_2(x)$ =38.8898

Population-II Source:[2]

It consists of 142 cities of India with population 100,000 and above; the character x and y being.

x : Census population in the year 1961.

y : Census population in the year 1971.

$\bar{Y}$ =4015.2183, $\bar{X}$ =2900.3872, C_Y =2.1118, C_X =2.1971, ρ =0.9948, $\beta_2(x)$ =48.1567.

Population-III Source:[1]

The variance are defined as follows

x : The number of persons per block

y : The number of rooms per block

$\bar{Y}$ =101.1, $\bar{X}$ =58.80, C_Y =0.14450, C_X =0.1281, ρ =0.6500, $\beta_2(x)$ =2.238

Population-IV Source:[3]

The population constants are as follows:

N=20, n=8, $\overline{Y}$=19.55, $\overline{X}$=18.8, C_Y^2=0.1262, C_X^2=0.1555, ρ=0.9199, $\beta_2(x)$=3.0613.

Table 13.1 Percent Relative Efficiencies of T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, $\overline{Y}_R$, $\overline{Y}_P$ and $\hat{\overline{Y}}_{SW}^{opt}$ w. r. t. $\overline{y}$.

Estimators	Pop. II	Pop. II	Pop. III	Pop. IV
$\overline{y}$	100	100	100	100
T_1	169.57	8077.28	158.09	*
T_2	199.32	8473.83	172.83	*
T_4	162.505	8052.06	158.99	*
T_5	*	*	*	550
T_6	169.57	*	148.14	*
T_7	*	*	*	582.02
T_8	*	*	*	465.30
$\overline{Y}_R$	8031.11	*	157.86	*
$\overline{Y}_P$	156.39	*	*	526.49
$\hat{\overline{Y}}_{SW}^{opt}$	208.45	9640.41	173.16	650.26

* datas are mot applcable.

Conclusion

Table 1 shows that there is a significant gain in efficiency by using the proposed asymptotically optimum estimator $\hat{\overline{Y}}_{SW}^{opt}$ over the unbiased estimator $\overline{y}$; T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, usual ratio estimator $\overline{Y}_R$ and product estimator $\overline{Y}_P$.

References

1. Cochran, W. G., Sampling Techniques. *John Wiley & Sons,* (2007).
2. Das, A. K. Contributions to the theory of sampling strategies based on auxiliary information, *Unpublished Ph.D. Thesis, Bidhan Chandra Krishi Vishwavidyalaya*, India, (1998).
3. Pandey, B. N. and Dubey, V., Modified product estimateor using coefficient of variation of auxiliary variate, *Assam Statistical Review*, 2(2), 64-66(1998).
4. Rao, P. S. R. S., and Mudholkar, G. S., Generalized multivariate estimator for the mean of finite populations, *J. of the American Statistical Association*, 62 (319), 1009-1012 (1967).
5. Robson, D. S., Applications of multivariate polykays to the theory of unbiased ratio-type estimation, *J. of American Statistical Association*, 52 (280), 511-522 (1957).
6. Singh, G. N., On the improvement of product metrhod of estimation in sample surveys, *J. of Indian Society of Agricultural Statistics*, 56 (3), 267-275 (2003).
7. Singh, H. P., and Espejo, M. R., On linear regression and ratio product estimation of a finite population mean, *J. of Royal Statistical Society*, 59-67(2003).
8. Singh, H. P., Tailor, R., and Kakran, M. S., An improved estimator of population mean using power transformation. *J. of the Indian Society of Agricultuaral Statistics*, 58 (2), 223-230 (2004).
9. Singh, H. P., and Tailor, R., Use of known correlation coefficient in estimating the finite population mean, *Statistics in transition*, 6(4), 555-560 (2003).
10. Sisodia, B. V. S., and Dwivedi, V. K., Modified ratio estimator using coefficient of variation of auxiliary variable, *Unknown journal* (1981).
11. Tailor, R., and Sharma, B. K., A modified ratio cum product estimator of finite population mean using known coefficient of variation and coefficient of kurtosis, *Statistics in Transition-new series*, 10(1), 15-24 (2009).
12. Upadhyaya, L.N., and Singh, H.P., Use of transformed auxiliary variable in estimating the finite population mean, *Biometrical Journal*, 41(5), 627-636 (1999).

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Some Categorical Aspects of Groups

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Abstract

An arbitrary group can be thought of as a category with one object in which every morphism is an isomorphism. In this paper we have shown how an arbitrary group can be considered as a category with one object. Also we have defined quotient category of a group. The categorical approach to the fundamental theorem of homomorphism of group theory has been provided. Moreover the isomorphism theorems of groups have been proved categorically.

Keywords: *Category; zero object; zero morphism; kernel; cokernel; quotient category.*

Introduction

Here we discussed some categorical aspects of Groups in details. We also provided categorical proof of isomorphism theorems of groups.

Preliminaries

For notions of category theory we shall in general follow the notationand terminology of Popescu [6]. However, we do deviate somewhat.

For **C** a category and A, B objects of **C**, Mor(A, B) denotes the set of morphisms from A to B.

It will be shown that an arbitrary group can be thought of as a ategory with one object in which every morphism is an isomorphism.

Next we shall use the definition of quotient category from

Mitchel [3] and **quotient category of a group** will be defined.

If G and H are groups, regarded as categories, then we can consider arbitrary functors between them $f : G \to H$. It is obvious that a functor between groups is exactly the same thing as a group homomorphism.

We will also think the fundamental theorem of homomorphism of group theory in categorical way.

Lastly the 2nd and 3rd isomorphism theorems of groups will be proved categorically by using the fact that "**every morphism in the category of groups (*Gp*) has a cokernel.**"

Main Results:

1. In the category of groups (*Gp*) every morphism has a kernel.

Proof: Let us consider the morphism $K \xrightarrow[i]{} G \xrightarrow[f]{} H$.

Let kerf $=$ K be the kernel of *f*.

Let us consider the diagram $K \xrightarrow[i]{} G \xrightarrow[f]{} H$, where *i* is inclusion map.

Clearly foi = **0**, $0 : K \to H$ being zero morphism.

Suppose that $g : M \to G$ be another morphism such that

fog = **0** (i)

Let us define $j : M \to K$ by $j(m) = g(m)$ for all $\Rightarrow y^{-1} a \in G$.

This is well defined as $f(g(m)) = (fog)(m) = \mathbf{0}(m) = e$ [from (i)]

$\Rightarrow g(m) \in K$

Now $(ioj)(m) = i(j(m)) = j(m) = g(m)$ for all $m \in M$.

$\Rightarrow ioj = g$

If $j' : M \to K$ be another morphism such that $ioj' = g$.

Then $ioj = ioj' = i(j(m)) = i(j'(m)) \ \ \forall\, m \in M$

$\Rightarrow j(m) = j'(m)$ [i is inclusion] $\Rightarrow j = j'$

Thus j is unique.

Hence $i : K \to G$ is a kernel of $f : G \to H$.

2. In the category of groups(*Gp*) every morphism has a cokernel.

Proof: Let $f : G \to H$ be morphism in ***Gp.*** Let us consider the diagram

$G \xrightarrow[f]{} H \xrightarrow[p]{} H/N$, where N is the smallest normal subgroup of H containing $f(G)$ i.e. normal closure of $f(G)$ in H and p is natural homomorphism.

Let us consider a morphism $g : H \to S$ such that $gof = \mathbf{0}$.

Let us define $j : H/N \to S$ by $j(hN) = g(h)$.

It is well defined as $xN = yN$ for $x, y \in H \Rightarrow y^{-1}x \in N \Rightarrow y^{-1}x$ is a finite product of elements of the form $d\, f(a)d^{-1}$,where $a \in G$ and $d \in H$.

Since $g(df(a)d^{-1}) = g(d)g(f(a))\, g(d^{-1}) = g(d)(gof)(a)g(d^{-1})$

$= g(d)\mathbf{0}(\mathrm{I})g(d^{-1}) = g(d)e_S\, g(d^{-1}) = g(dd^{-1}) = e_S$

Thus $g(y^{-1}x) = e_S \Rightarrow g(y^{-1})g(x) = e_S \Rightarrow g(x) = g(y) \Rightarrow j(xN) = j(\mathrm{I}N)$.

Now $(jop)(h) = j(p(h)) = j(hN) = g(h)$ for all $h \in H$.

$\Rightarrow jop = g$.

Also j is unique as p is epimorphism.

Hence $p : H \to H/N$ is a cokernel of $f : G \to H$.

3. A Group can be thought of as a category with one object:

Let us consider an arbitrary group (G , .).

Let us consider the collection G' as follows-------------

i) $ObG' = \{G\}$

ii) The only set **Mor(G,G)** and the morphisms are the elements of G

i.e. $g \in G \Leftrightarrow g : G' \to G$.

iii) The composition in **Mor(G,G)** is defined as— ,

if $x : G' \to G$, $y : G' \to G$ then $yox : G' \to G$ is defined as $yox = y.x$

Then we have the following

a) $zo(yox) = zo(y.x) = z.(y.x) = (z.y).x = (zoy)ox$

Hence "o" is associative.

b) let "e" be the identity element in G i.e. $e : G' \to G$ and for $x : G' \to G$

$y : G' \to G$ we have $xoe = x.e = x$ and $eoy = y$.

Therefore $e : G' \to G$ is the identity morphism in **Mor(G,G)**. (we shall frequently write 1_G for $e : G' \to G$)

Hence G' is a category.

It is clear that every morphism in G' is isomorphism. Because for any x: $G \to G$ we have a unique x^{-1}: $G \to G$ such that $xox^{-1} = x^{-1}ox = 1_G$.

Here onwards we call the category corresponding to the group G as G'.

4. Quotient Category of a Grouop

Let N be a normal subgroup of G.

Let us define a relation "$\approx$" in **Mor (G,G)** as follows—-

For any x,y $\in$ **Mor (G,G)**,

$x \approx y \Leftrightarrow xy^{-1} \in N$.

Now we shall prove that R is a congruence relation on **Mor**(G,G).

i) $x \approx x \Leftrightarrow xx^{-1} = e \in N$.

ii) Let $x \approx x \Rightarrow xy^{-1} \in N \Rightarrow (xy^{-1})^{-1} \in N \Rightarrow yx^{-1} \in N \Rightarrow y \approx x$.

iii) Let $x \approx y$ and $y \approx z$ then $xy^{-1} \in N$ and $yz^{-1} \in N$

$\Rightarrow (xy^{-1})(yz^{-1}) \in N \Rightarrow x(y^{-1}y)z^{-1} \in N \Rightarrow xe\, z^{-1} \in N$

$\Rightarrow xz^{-1} \in N \Rightarrow x \approx z$.

Thus $\approx$ is an equivalence relation.

Next, let $x \approx y$ and $a \approx b \Rightarrow xy^{-1} \in N$ and $ab^{-1} \in N$

Now $(xa)(yb)^{-1} = (xa)(b^{-1}y^{-1}) = x(y^{-1}y)ab^{-1}y^{-1} = (xy^{-1})(y(ab^{-1})y^{-1}) \in N$

$\Rightarrow xa \approx yb$.

Also $x \approx y$ and $c \approx d \Rightarrow xy^{-1} \in N$ and $cd^{-1} \in N$

Now $(cx)(dy)^{-1} = (cx)(y^{-1}\, d^{-1}) = c(d^{-1}d)x\, y^{-1}\, d^{-1} = (cd^{-1})(d\,(xy^{-1})\, d^{-1}) \in N$

$\Rightarrow cx \approx dy$

Thus $\approx$ is a congruence on **Mor** (G,G).

Next we define quotient category $G'/\approx Q_{G'}$ of G' as followsi)

Ob($Q_{G'}$) = **Ob**(G'),

ii) **Mor(**$Q_{G'}$) = {the equivalence classes E(x) : x **Mor** (G,G)}

where E(x)={y **Mor**(G,G)| $y \approx x$ }.

Let us define composition in Mor($Q_{G'}$) as E(x)oE(y)=E(xoy), which is well defined as

If $E(x) = E(a)$ and $E(y) = E(b)$

then $x \approx a$ and $y \approx b$.

$\Rightarrow xa^{-1} \in N$ and $yb^{-1} \in N$.

Now, $(xy)(ab)^{-1} = xyb^{-1}a^{-1} = x(a^{-1}a)yb^{-1}a^{-1} = (xa^{-1})\{a(yb^{-1})a^{-1}\} \in N$

$\Rightarrow xy \approx ab \Rightarrow E(xoy) = E(aob)$.

5. Categorical approach to the fundamental theorem of Homomrphism of Group theory:

Let *f*: *G*à *H* be a homomorphism of the group G on to the group *H*.

Then *N*=ker*f* is a normal subgroup of *G*.

Then obviously $f : G' \rightarrow H'$ will be a full functor which is surjective on object *S*.

Let us consider the quotient category $Q_{G'}$ of G'.

Let us define $F: Q_{G'} \rightarrow H'$ by

$F(G) = H$ and

$F(E(x)) = f(x)$ where x: GàG and $f(x)$: Hà H, which is well defined as

If $E(x)=E(y)$ then $x \approx y \Rightarrow xy^{-1} \in N \Rightarrow f(xy^{-1}) = e_H$

$\Rightarrow f(x)f(y^{-1}) = e_H \Rightarrow f(x) = f(y)$

Now,

i) $F(E(y)oE(x)) = F(E(yox)) = f(y.x) = f(y).f(x) = F(E(y))oF(E(x))$.

ii) $F(E(1_G)) = F(E(e)) = f(e) = 1_H = 1_{F(G)}$

F is a covariant functor.

Conversely, let us define $F: H' \rightarrow Q_{G'}$ by

$D(H) = G$, $D(f(x))=E(x)$, which is well defined as $N=\ker f$ is normal subgroup of G.

Now,

i) $D(f(y)of(x)) = D(f(yox)) = D(f(y.x)) = E(y.x) = E(y)oE(x)$

$= D(f(y))$ o $D(f(x))$.

ii) $D(1_H) = D(f(e)) = E(e) = 1_{D(H)}$.

G is covariant functor.

Thus $FoD(f(x)) = F(D(f(x))) = F(E(x)) = f(x) = Id_{H'}(f(x))$.

$FoD = Id_{H'}$.

Similarly it can be proved that $DoF = IdQ_{G'}$.

Hence $Q_{G'} \cong H'$.

6. Lemma:

Let $f: G \rightarrow H$ be a group homomorphism such that f kills N (i.e. $f(N)=e_H$) where N is a normal subgroup of G. Then there exists a unique homomorphism $f': G/N \rightarrow H$ with $f'op = f$, where $p: G \rightarrow G/N$ is canonical homomorphism i.e the following diagram

$G \xrightarrow[f]{} H$, $G \xrightarrow[P]{} G/N$, $G/N \xrightarrow[f']{} H$ commutes.

Proof: Let N be a normal subgroup of G.

Then we have

$N \xrightarrow[i]{} G \xrightarrow[P]{} G/N$ [where the elements of G/N are the equivalence classes of the form $E(g)$ for all $g \in G$ and $p: G \to G/N$ is canonical homomorphism and $i: N \to G$ is inclusion]

such that p o i = u , where $u: N \to G/N$ is zero homomorphism as,

$(poi)(n) = p(i(n)) = p(n) = N$ = zero element in $G/N = u(n)$.

$\Rightarrow poi = u$.

Let $f: G \to H$ be a group homomorphism such that

$foi = u$ i.e. $(foi)(n) = u(n)$ for all $n \in N$

$\Rightarrow f(i(n)) = e_H \Rightarrow f(n) = e_H \Rightarrow f(N) = e_{H.} \Rightarrow f$ kills N.

Next let us define $f': G/N \to H$ by $f'(E(g)) = f(g)$.

which is well defined as if E(g) = E(h)

then $g \approx h \Rightarrow gh^{-1} \in N \Rightarrow f(gh^{-1}) = e_H$ (since f kills N)

$\Rightarrow$ i(g) $f(h^{-1}) = e_H \Rightarrow f(g) = f(h)$.

Also $(f'op)(g) = f'(p(g)) = f'(E(g)) = f(g) \Rightarrow f'op = f$.

Suppose ,if possible, $f'': G/N \to H$ be another homomorphism such that $f''op = f$.

Then $f''op = f'op \Rightarrow f'' = f'$ (since p is surjective).

Hence f' is unique.

7. Categrical proof of Isomrphism theorems of groups :

Theorem 1: Let M, N are normal subgroups of G such that $M \subseteq N$. Then $G/M/N/M \cong G/N$.

<u>Proof</u> : As N is a normal subgroup of G so $p_N: G \to G/N$ is a

group homomorphism and it kills M (since $M \sqsubseteq N$). Therefore by the ***lemma6*** we have a unique homomorphism $f: G/M \to G/N$ such that the following diagram

$$G \xrightarrow[p_N]{} G/N\,,\ G \xrightarrow[p_M]{} G/M,\ G/M \xrightarrow[f]{} G/N \text{ commutes i.e.}$$

$$fop_M = p_N \qquad \text{(i)}$$

Now $f: G/M \to G/N$ is a group homomorphism which kills N/M, therefore by **lemma 6** there exists a unique homomorphism

$f': G/M/N/M \to G/M$ such that the following diagram

$$G/M \xrightarrow[f]{} G/N,\ G/M \xrightarrow[p_{N/M}]{} G/M/N/M,$$

$$G/M/N/M \xrightarrow[f']{} G/N \text{ commutes i.e. } f'op_{N/} = f \qquad \text{(ii)}$$

But we have also

$$G \xrightarrow[P_M]{} G/M \xrightarrow[P_{N/M}]{} G/M/N/M,$$

$$G \xrightarrow[P_{N/M}oP_M]{} G/M/N/M \text{ which kills N.}$$

So we have a unique homomorphism $k: G/N \to G/M/N/M$ such that the following diagram

$$G \xrightarrow[P_{N/M}oP_M]{} G/M/N/M,\ G \xrightarrow[P_N]{} G/M,$$

$$G/N \xrightarrow[k]{} G/M/N/M \text{ commtes i.e. } kop_N = p_{N/M}op_M$$

From (ii) we have k o f' o $p_{N/M}$ o p_M = k o f o p_M = k o p_N [from (i)]

= $p_{N/M}$ o p_M [from (iii)]

Therefore ko f' = $Id_{G/M/N/M}$ (iv)

Similarly from (iii) we have f' o k o p_N = f' o $p_{N/M}$ o p_M

= f o p_M [from (ii)]

= p_N [from (i)]

Thus $f' \circ k = Id_{G/N.}$ (v)

From (iv) and (v) we have $G/N \cong G/M/N/M$.

Theorem 2 : Let $H, N \leq G$ and N is normal subgroup of G. Then $HN/N \simeq H/H\cap N$.

Proof : As N is normal subgroup of G so it is normal in HN.

So we may compose the inclusion $i : H \rightarrow HN$ with the natural homomorphism $p'' : HN \rightarrow HN/N$ to get a homomorphism

$g : H \rightarrow HN / N$ [i.e. $p'' \circ i = g$].

It kills HN. Therefore by **lemma 6** we have a unique homomorphism $f : H/H\cap N \rightarrow HN/N$ such that the following diagram

$H \xrightarrow[g]{} HN/N$, $H \xrightarrow[p']{} H/H\cap N$,

$H/H\cap N \xrightarrow[f]{} HN/N$ commutes i.e. $f \circ p' = g$ (i)

Also $g' : HN \rightarrow H/H\cap N$ is a group homomorphism which kills N.

So by the **lemma 6** we have a unique homomorphism $f' : HN/N \rightarrow H/H\cap N$ such that the following diagram g'

$HN \rightarrow H/H\cap N$, $HN \xrightarrow[p'']{} HN/N$, $HN/N \xrightarrow[f']{} H/H\cap N$

commutes i.e. $f' \circ p'' = g'$ (ii)

So from (i) we have $f' \circ f \circ p' = f' \circ g = p'$

Therefore $f' \circ f = Id_{H/HN}$ (iii)

Also from (ii) we have $f \circ f' \circ p'' = f \circ g' = p''$

Thus $f \circ f' = Id_{HN/N}$ (iv)

From (iii) and (iv) we have

$$HN/N \simeq H/H\cap N.$$

7. Regarding a group G as a category with one object, a categorical congruence $\sim$ on G is the same thing as a normal subgroup $N \subseteq G$, that is, the two kinds of things are isomorphic correspondence.

Solution: Let G be a group and $f : G \to H$ be a group homomorphism.

Let us consider the relation defined by

$g \sim h \Leftrightarrow f(g) = f(h)$ for all $g, h \in G$.

we shall prove that this is an equivalence relation.

(i) As $f(g) = f(g)$ for all $g \in G$

Therefore $g \sim g \Rightarrow \approx$ is reflexive.

(ii) Let $g \sim h$ then $f(g) = f(h) \Rightarrow f(h) = f(g) \Rightarrow h \sim g \Rightarrow \sim$ is symmetric.

(iii) Let $g \sim h$ and $h \sim k$ then $f(g) = f(h)$ and $f(h) = f(k)$

$\Rightarrow f(g) = f(k) \Rightarrow g \sim k \Rightarrow \sim$ is transitive.

Thus $\sim$ is an equivalence relation.

Now, because f is a group homomorphism so

$f(g) = f(h)$ and $f(g') = f(h')$ implies that $f(gg') = f(g) f(g') = f(h) f(h') = f(hh')$ which gives an additional axiom:

(iv) **Compatibility with multiplication**: If $g \sim h$ and $g'\ h' \Rightarrow gg'\ hh'$

This defines a congruence relation on G.

Let us denote the collection of all equivalence classes be $G/\sim$.

Compatibility with multiplication is precisely the condition needed for multiplication in G to be well defined on the equivalence classes $G/\sim$.

So given a congruence relation on group we can get a quotient map $p: G \to G/\sim$, which is a group homomorphism.

However due to inverses compatibility with multiplication shows that

$g \sim h \Leftrightarrow f(g) = f(h) \Leftrightarrow f(h^{-1}g) = f(e) \Leftrightarrow h^{-1}g$.

Also $g \sim h \Leftrightarrow f(g) = f(h) \Leftrightarrow f(gh^{-1}) = f(e) \Leftrightarrow gh^{-1}$

Thus in other words, a congruence relation is completely determined by which elements are congruent to the identity. Let us denote the set of these elements be N. Then

(i) N having an identity is equivalent to $\sim$ being reflexive.

(ii) N being closed under multiplication is equivalent to $\sim$ being transitive.

(iii) N being closed under inverses is equivalent to being symmetric.

Thus we can say that $\sim$ is an equivalence relation iff N is a subgroup.

Also from compatibility with multiplication we have that

$$g \sim e \Leftrightarrow hg \sim h \Leftrightarrow hgh^{-1} \sim e.$$

It follows that N has another property that it is closed under conjugation, so it is a normal subgroup of G.

Conversely if we define an equivalence relation by

$g \sim h \Leftrightarrow h^{-1}g \in N$, where N is normal subgroup of G.

Then $g \sim h \Leftrightarrow h^{-1}gN$ and $g' \sim h' \Leftrightarrow h'^{-1} g' \in N$

and hence $h'^{-1}h^{-1}g\,g' = h'^{-1}h^{-1}\,g(h'\,h'^{-1})\,g'$

$$= \{h'^{-1}({}^{1}g)\,h'\}\,h'^{-1}\,g' \in N \Rightarrow g\,g' \sim h\,h'.$$

So normality of N is equivalent to $\sim$ being compatible with multiplication i.e. $\sim$ being congruence relation. Hence the result.

Conclusions

In this paper we have used some categorical notions to prove some theorems of group theory.The works have been done so far can be extended to the ring theory also.

References

1. Anderson, F. W., and Fuller, K. R., *Rings and Categories of Modules*, Springer-Verlag, New York, Berlin, Heidelberg, London, paris, Tokyo, Hong Kong. Barcelona, Budapast.

2. Mac Lane, S., *Categories for the Working Mathematician*, Springer-Verlag New York Berlin (1971).

3. Mitchel, B., *Theory of Categories*, Academic Press New Yorkand London (1965).

4. Krishnan,V. S., *An introduction to Category Theory*, North Holland NewYork Oxford (1981).

5. Schubert, H., *Categories*, Springer-Verlag Berlin Heidelberg New York (1972).

6. Popescu,N., *Abelian Categories with Applications to Rings and Modules,* Academic Press, London & New York (1973).

7. Awodey, S., *CategoryTheory,Second Edition* Clarendon Press,Oxford (2006).

8. Borceux, F., *Hand Book of Categorical Algebra*, Cambridge University Press (1994).

9. Simmons, H., *An Introduction to Category Theory,* Cambridge University Press (2011).

10. Freyd, P., *Abelian Categories, An Introduction totheTheory of Functors,* A Harper International Edition, jointly published by Harper & Row,NewYork,Evaston & London and John Weatherhill Inc. Tokyo (1965).

11. Pareigis,Bodo., *Categories and Functors,* Academic Press New York, London (1970).

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Semi Analytical Solution of DGLAP Evolution Equation up to Next-to-Leading Order for Non-Singlet Spin Structure Function using a Q^2 Dependent ReggeAnsatz

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Abstract

Using a Q^2 dependent Reggeansatz for non-singlet part of spin dependent structure function $g_1^{NS}(x, Q^2)$ as the initial input, we solve Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations in leading order and next-to-leading order. The solutions which govern the Q^2 evolution of structure functions are analyzed phenomenologically in comparison with available results taken from different experiments and parametrizations. Considerable success achieved in these regards reflects the capability of the Reggeansatz in evolving spin structure functions with and Q^2 in accord with DGLAP equations at small- .

***Key Words**: Regge behaviour of structure function, DGLAP evolution equation, Spin structure function.*

AMS Subject Classification No (2010): 33F05, 35R09

Introduction

Deep inelastic scattering (DIS) of leptons (electron, muon, neutrino) from nucleon is recognized as an important testing ground for investigation of the structure of nucleon. Spin is a fundamental property of nucleon and the spin structure of nucleon has been one of the most active frontiers over the last two decades. How the spin of the nucleon is distributed among its constituent partons (quarks, gluons) is the key question. The distribution of spin among quarks and gluons are represented by the spin dependent structure function $g_1(x, Q^2)$ and the determination and understanding of the shape of this function have been an important issue[1].

In Quantum Chromodynamics (QCD), the spin structure function $g_1(x, Q^2)$ is described as Mellin convolutions between parton distribution functions $(\Delta q_i, \Delta g)$ and the Wilson coefficients C_i[2]

$$g_1(x, Q^2) = \frac{1}{2n_f}\sum_{i=1}^{n} e_i^2 \left[C_{NS}\otimes\Delta q_{NS} + C_S\otimes\Delta q_S + 2n_f C_g\otimes\Delta g\right],$$

Which consists of three parts, non-singlet

$(g_1^{NS}(x, Q^2) = \frac{1}{2n_f}\sum_{i=1}^{n} e_i^2\, [C_{NS}\otimes\Delta q_{NS}])$, singlet

$(g_1^{S}(x, Q^2) = \frac{1}{2n_f}\sum_{i=1}^{n} e_i^2\, [C_{S}\otimes\Delta q_{S}])$ and gluon

$(G(x, Q^2)\frac{1}{2n_f}\sum_{i=1}^{n} e_i^2\, [2n_f C_g\otimes\Delta g\,])$. The Q^2 distribution of these spin dependent non-singlet, singlet and gluon distribution functions are governed by a set of integro-differential equations, theDokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equationswhich are given by[3-6]

$$Q^2\frac{dxg_1^{NS}(x, Q^2)}{d\ln Q^2} = \frac{\alpha(Q^2)}{2\pi}P_{qq}^{NS}(x, Q^2)\otimes xg_1^{NS}(x, Q^2) \tag{1}$$

$$Q^2\frac{d}{d\ln Q^2}\begin{pmatrix} g_1^S(x, Q^2) \\ \Delta G(x, Q^2)\end{pmatrix} = \frac{\alpha(Q^2)}{2\pi}\begin{pmatrix} P_{qq}^S(x, Q^2) & 2n_f P_{qg}^S(x, Q^2) \\ P_{gq}^S(x, Q^2) & P_{gg}^S(x, Q^2)\end{pmatrix}\otimes\begin{pmatrix} g_1^S(x, Q^2) \\ \Delta G(x, Q^2)\end{pmatrix} \tag{2}$$

Here P_i are the polarized splitting functions [6-8]. These equations are valid to all orders in the strong coupling constant $\frac{\alpha(Q^2)}{2\pi}$.

Although QCD predicts the Q^2 dependence of structure functions in accord with the DGLAP equations but they have limitations on absolute predictions of structure functions. DGLAP equations can only predict the evolution of structure functions with Q^2, once an initial distribution is given. In order to solve DGLAP equation there are different methods available in literature (**see the references in recent article [9]**). The DGLAP equations can be solved exactly in moment space [10], provided a set of input distributions is specified at an initial value Q^2 and by inverting the moments Q^2 the structure functions can be obtained. However, this method requires the knowledge of initial distributions of structure functions at all values of x from 0 to 1, and no experimental measurements at fixed Q^2 can reach all the way to $x = 0$. Therefore, in current analysis this set of equations are solved numerically by using an initial input distribution of the structure function at a fixed Q^2, in terms of some free parameters, the parameters are so adjusted that the parametrization best fit the existing data. Although many parameterizations are available in literature (**see the references in recent article [9]**) in order to predict the initial distribution of structure functions, but most of them are with several parameters. Again a parametrization with large number ofparameters creates difficulties in obtaining best fitting and hence leads towards inaccuracy in results. Therefore explorations of the possibility of obtaining accurate solutions of DGLAP evolution equations with less number of parameters are always interesting.

In this paper we have discussed a semi analytical way of solving DGLAP equations using a simple Q^2 dependent Regge ansatz for the non-singlet part of spin dependent structure function $g_1^{NS}(x,Q^2)$as the initial input given by

$$xg_1^{NS}(x,Q^2) = A.x^{1-b\ln\frac{Q^2}{\Lambda^2}} = A.x^{1-bt}, \tag{3}$$

where $t = \text{In}\frac{Q^2}{\Lambda^2}$ and Λ is the QCD coupling constant. Here A and b are two fitting parameters. However in accordance with Ref. [9], is the only parameter to be obtained by fitting the existing data. As the model consists of only one parameter, which can be easily fitted and hence there is no possibility of inaccurate result.In our fitting analysis for this parametrization with the experimental data within the kinematical range$x$ < 0 and $Q^2 = 5.0\ GeV^2$, the best fitting value of the parameter is found to be $b = 0.06915 \pm 0.017$. The reason behind consideration of this type of model for $xg_1^{NS}(x, Q^2)$ structure function lies in the fact that the $g_1^{NS}(x, Q^2)$ is a Regge behaved structure function[11], i.e., for $x << 1$, $g_1^{NS}(x, Q^2)$ rises approximately like a power of x towards low x. This type of small x behavior is also observed in case of the non-singlet structure function $xF_3(x, Q^2)$. Again, the Q^2 behaviour for both $xg_1^{NS}(x, Q^2)$ and $xF_3(x, Q^2)$ are governed by the same DGALP equation. As both x and Q^2 behaviour of these two structure functions are similar and $xF_3(x, Q^2)$ satisfies the behaviour (3)[9], therefor we have considered the possibility of $xg_1^{NS}(x, Q^2)$ to be consistent with the behaviour (3). Thus considering the model (3) to be satisfied by $xg_1^{NS}(x, Q^2)$ structure function, we have solved the DGLAP equation in leading order(LO) and next-to-leading order (NLO) and performed a phenomenological analysis of our results in comparison with other experimental results. We observe a very good agreement between our results with that of experiments which reflects that $xg_1^{NS}(x, Q^2)$ is consistent with the ansatz (3).

Solution of DGLAP evolution equation in LO and NLO

The DGLAP equation (2) for structure function can be expressed in terms of the variable as

$$\frac{dxg_1^{NS}(x,t)}{dt} = \frac{\alpha(t)}{2\pi}\int_x^1 \frac{d\omega}{\omega} xg_1^{NS}\left(\frac{x}{\omega}, t\right) P_{qq}^{NS}(\omega) \tag{4}$$

Here the splitting function $P_{qq}^{NS}(\omega)$ is defined up to next-to-leading order by

$$P_{qq}^{NS}(\omega) = \frac{\alpha(t)}{2\pi} P^0(\omega) + \left(\frac{\alpha(t)}{2\pi}\right)^2 P^1(\omega) + \cdots ...$$

where, $P^0(\omega)$ and $P^1(\omega)$ are the corresponding leading order and next-to-leading order corrections to the splitting functions[6-8]. Again, in LO and NLO, the coupling constant $\alpha(t)/2\pi$ is given by [12] $\left(\frac{\alpha(t)}{2\pi}\right)_{LO} = \frac{2}{\beta_0 t}$ and $\left(\frac{\alpha(t)}{2\pi}\right)_{NLO} = \frac{2}{\beta_0 t}\left[1 - \frac{\beta_1 \ln t}{\beta_0^2 t}\right]$ respectively, where β_0 and β_1 are the one loop and two loop corrections of QCD β function. Substituting the respective splitting functions along with the corresponding running coupling constant in (4), the DGLAP evolution equations in LO and NLO become

$$\frac{dxg_1^{NS}(x,t)}{dt} = \left(\frac{\alpha(t)}{2\pi}\right)_{LO}\left[\frac{2}{3}\{3+4\ln(1-x)\}xg_1^{NS}(x,t) + I_1(x,t)\right] \quad (5)$$

$$\frac{dxg_1^{NS}(x,t)}{dt} = \left(\frac{\alpha(t)}{2\pi}\right)_{NLO}\left[\frac{2}{3}\{3+4\ln(1-x)\}xg_1^{NS}(x,t) + I_1(x,t)\right] + \left(\frac{\alpha(t)}{2\pi}\right)_{NLO}^2 I_2(x,t) \quad (6)$$

respectively. Here the integral functions are given by,

$$I_1(x,t) = \frac{4}{3}\int_x^1 \frac{d\omega}{(1-\omega)}\left[(1+\omega^2).\frac{x}{\omega}g_1^{NS}\left(\frac{x}{\omega},t\right) - 2xg_1^{NS}(x,t)\right]$$

and $I_2(x,t) = \int_x^1 \frac{d\omega}{\omega} P^1(\omega)\frac{x}{\omega}g_1^{NS}\left(\frac{x}{\omega},t\right)$.

We now solve the DGLAP evolution equations in LO and NLO considering the Regge likeansatz, $xg_1^{NS}(x,t) = A.x^{1-bt}$ as the initial input. Here firstly we will discuss the solution of DGLAP equations in LO explicitly and then extend the same formalism to obtain the solutions for the case of NLO. Thus, substituting $xg_1^{NS}(x,t) = A.x^{1-bt}$ and $\frac{x}{\omega}g_1^{NS}\left(\frac{x}{\omega},t\right) = A.\left(\frac{x}{\omega}\right)^{1-bt}$ in equation (5) we get

$$\frac{dxg_1^{NS}(x,t)}{dt} = \left(\frac{\alpha(t)}{2\pi}\right)_{LO}\left[\frac{2}{3}\{3+4\ln(1-x)\} + \frac{4}{3}\int_x^1 \frac{d\omega}{(1-\omega)}[(1+\omega^2).\omega^{bt-1} - 2]\right]xg_1^{NS}(x,t)$$

This equation can easily be solved to have

$$xg_1^{NS}(x,t) = C \exp\left[\int \left(\frac{\alpha(t)}{2\pi}\right)_{LO} P(x,t)\, dt\right] \tag{7}$$

w i t h $P(x,t) = \frac{2}{3}\{3 + 4\ln(1-x)\} + \frac{4}{3}\int_x^1 \frac{d\omega}{(1-\omega)}[(1+\omega^2).\omega^{bt-1} - 2]$,

and C is a constant originated due to integration. The expression (7) and gives the t dependence of $xg_1^{NS}(x, Q^2)$ structure function at fixed x. Let the fixed value for x be $x = x_0$.

Now, defining an input point

$$x_0 g_1^{NS}(x_0,t_0) = C \exp\left[\int \left(\frac{\alpha(t)}{2\pi}\right)_{LO} P(x_0,t)\, dt\right]\Bigg|_{t=t_0} \tag{8}$$

at any $t = t_0$ and dividing (7) by (8) and rearranging a bit we obtain the t evolution of $xg_1^{NS}(x,t)$ structure function with respect to the input point $xg_1^{NS}(x_0, t_0)$ as

$$x_0 g_1^{NS}(x_0,t) = x_0 g_1^{NS}(x_0,t_0) \exp\left[\int_{t_0}^{t} \left(\frac{\alpha(t)}{2\pi}\right)_{LO} P(x_0,t)\, dt\right] \tag{9}$$

Proceeding in the similar way, the DGLAP equation in NLO can be solved to have the solution representing the t evolution of $xg_1^{NS}(x,t)$ structure function as

$$x_0 g_1^{NS}(x_0,t) = x_0 g_1^{NS}(x_0,t_0)\, exp\left[\int_{t_0}^{t} \left(\frac{\alpha(t)}{2\pi}\right)_{NLO} P(x_0,t)\, dt + \int_{t_0}^{t} \left(\frac{\alpha(t)}{2\pi}\right)_{NLO}^{2} Q(x_0,t)\, dt\right], \tag{10}$$

where $Q(x,t) = \int_x^1 \frac{d\omega}{\omega} P^1(\omega)\omega^{-(1-bt)}$.

Substituting the expressions for strong coupling constant in LO and NLO in the corresponding expressions (9) and (10) respectively and performing integrations, we will obtain the t or Q^2 evolution of $xg_1^{NS}(x,t)$ structure function.

Again, as the initial parametrization (3) is assumed to predict both x and t dependence of $xg_1^{NS}(x,t)$ structure function, we may use it to obtain the x evolution of $xg_1^{NS}(x,t)$ with respect to an input point at for all as

$$xg_1^{NS}(x,t) = x_0 g_1^{NS}(x_0,t)\left(\frac{x}{x_0}\right)^{1-bt}$$

Substituting $x_0 g_1^{NS}(x_0,t)$ in the above equation from (9) and (10) we can obtain both x and t evolution of $xg_1^{NS}(x,t)$ structure function in LO and NLO as

$$xg_1^{NS}(x,t) = xg_1^{NS}(x_0,t_0)\exp\left[\int_{t_0}^{t}\left(\frac{\alpha(t)}{2\pi}\right)_{LO} P(x_0,t)\,dt\right]\left(\frac{x}{x_0}\right)^{1-bt} \quad (11)$$

and

$$xg_1^{NS}(x,t) = x_0 g_1^{NS}(x_0,t_0)\exp\left[\int_{t_0}^{t}\left(\frac{\alpha(t)}{2\pi}\right)_{NLO} P(x_0,t)\,dt + \int_{t_0}^{t}\left(\frac{\alpha(t)}{2\pi}\right)^2_{NLO} Q(x_0,t)\,dt\right]\left(\frac{x}{x_0}\right)^{1-bt} \quad (12)$$

respectively.

Results and Discussion

The $xg_1^{NS}(x,t)$ structure functions are calculated using the expressions (11) and (12) and their x and Q^2 evolutions are depicted in fig. 1(a) and fig. 1(b) respectively. The calculations, in this paper are made by using MATHEMATICA 9. In order to obtain sufficient numerical accuracy caution has been taken in evaluation of integrals containing higher order polynomials. The evaluations have been performed using Horner's method and it has improved speed and stability for numeric evaluation of large polynomials present in the splitting functions. In fig. 1(a) the x evolution of $xg_1^{NS}(x,t)$ structure functions are plotted in comparison with other experimental results taken from taken from SMC, E143, HERMES, and COMPASS experiments [13-18]. Here the input point about which the structure function is evolved is considered $x_0 g_1^{NS}(x_0,t_0) = 0.0133075$ at $x_0 = 0.0143955$ and $Q_0^2 = 5.0\ GeV^2$. However, as there are not any available experimental results for different Q^2, we could not have comparative analysis of our

Q^2 evolution results.Form the figures, we would like to emphasize that our results for the x dependence $xg_1^{NS}(x,t)$ structure function are in a good qualitative agreement with that of differentexperimental results. These agreement suggest that the Reggelike model $xg_1^{NS}(x,t) = A.x^{1-bt}$ with Q^2 dependent intercept is an efficient model in solving DGLAP equationin order to evolve structure functions at small x.

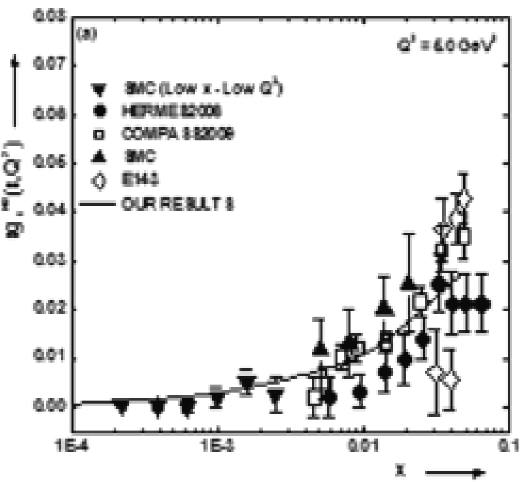

Fig. 15.1. (a) *x*-evolution

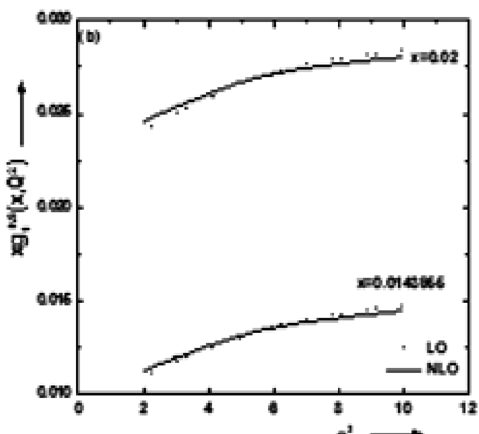

Fig. 15.1. (b) Q^2 evolution of $xg_1^{NS}(x,t)$ structure function

Conclusion

Here we have investigated how the initial input $xg_1^{NS}(x,t) = A.x^{1-bt}$, for the non-singlet part of spin structure function $xg_1^{NS}(x,t)$ helps in solving DGLAP equations up to NLO and obtain their x and Q^2 evolutions of $xg_1^{NS}(x,t)$. The structure functions, evolved as the solutions of the DGLAP equations are studied phenomenologically in comparison with the results taken from SMC, E143, HERMES, and COMPASS experiments. We observe a very good agreement between our theoretical results and other experimental results. The phenomenological success achieved in this study suggests that the QCD featured Regge behaved ansatz $xg_1^{NS}(x,t) = A.x^{1-bt}$ is capable of evolving $xg_1^{NS}(x,t)$ structure function in accord with DGLAP equations at small-x.

References

1. F. Taghavi-Shahri, F Arash, Phys. Rev. C 82, 035205 (2010)
2. G. Altarelli, R. D. Ball, S. Forte and G. Ridolfi, Nucl.Phys. B 496, 337 (1997)
3. V. N. Gribov and L N Lipatov, Sov. J. of Nucl. Phys., 15, 438 (1972)
4. L. N. Lipatov, Sov. J. of Nucl. Phys., 20, 94 (1975)
5. Y. L. Dokshitzer, Sov. Phys. JETP, 46, 641 (1977)
6. G. Altarelli and G. Parisi, Nucl. Phys. B, 126, 297 (1977)
7. R. Mertig and W. L. van Neerven, Z. Phys. C 70, 637 (1996)
8. W. Vogelsang, Phys. Rev. D 54, 2023 (1996)
9. N. M. Nath, N. Baruah and J. K. Sarma, Indian J. Phys. (2014) (DOI 10.1007/s12648-014-0467-9)
10. E, Reya, Phys. Rep. 69, 195 (1981)
11. J. Kwiecinski, Acta Phys. Pol. B 29(5), 1201 (1998)
12. B.G. Shaikhatdenov, A.V. Kotikov, V.G. Krivokhizhin, G. Parente, Phys. Rev. D 81, 034008 (2010)
13. M. G. Alekseev (COMPASS Collaboration), Phys. Lett. B 690, 466 (2010)
14. A. Airapetian*et al.* (HERMES Collaboration), Phys. Rev. D 75, 012007 (2007)
15. M. G. Alekseev (COMPASS Collaboration), Phys. Lett. B 660, 458 (2008); V. Yv. Alexakhin*et al.*, *ibid.* 647, 8 (2007)

16. B. Adeva*et al.* (SMC Collaboration), Phys. Rev. D 58, 112001 (1998)
17. B. Adeva*et al.* (SMC Collaboration), Phys. Rev. D 60, 072004 (1999)
18. K. Abe *et al.* (E143 Collaboration), Phys. Rev. Lett. 75, 25 (1995); Phys. Rev. D 58, 112003 (1998).