2 SEM TDC MTMH (CBCS) C 3

2022

(June/July)

MATHEMATICS

(Core)

Paper : C–3

(Real Analysis)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Define ε-neighbourhood of a point. 1

(b) Find the infimum and supremum, if it exists for the set $A = \{x \in \mathbb{R} : 2x + 5 > 0\}$. 2

(c) If

$$S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

then show that inf $S = 0$, where inf S denotes the infimum of S. 3

(d) State and prove that Archimedean Property of real numbers. 4

(e) Let $S \subseteq \mathbb{R}$ be a set that is bounded above and for $a \in \mathbb{R}$, $a + S$ is defined as $a + S = \{a + s : s \in S\}$. Show that $\sup(a + S) = a + \sup(S)$, where $\sup(S)$ denotes the supremum of S. 5

2. (a) State the Completeness Property of real numbers. 1

(b) Show that

$$\sup\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 1$$ 2

(c) Let

$$I_n = \left[0, \frac{1}{n}\right]$$

for $n \in \mathbb{N}$. Prove that

$$\bigcap_{n=1}^{\infty} I_n = 0$$ 3

(d) Prove that the set of real numbers is not countable. 4

Or

If

$$S = \left\{ \frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N} \right\}$$

find $\inf S$ and $\sup S$.

(e) State and prove the nested interval property. 5

Or

Prove that there exists a real number x such that $x^2 = 2$.

3. (a) State the Monotone Subsequence Theorem. 1

(b) Show that

$$\lim_{n \to \infty} \left(\frac{n}{n^2 + 1} \right) = 0$$ 2

(c) Show that a convergent sequence of real numbers is bounded. 3

(d) Show that

$$\lim_{n\to\infty}(b^n)=0$$

if $0<b<1$. 4

Or

Show that

$$\lim_{n\to\infty}(c^{\frac{1}{n}})=1$$

for $c>1$.

(e) State and prove the Monotone Convergence theorem. 5

Or

Let $Y:=(y_n)$ be defined as $y_1=1$, $y_{n+1}=\frac{1}{4}y_n+2$, $n\geq 1$. Show that (y_n) is monotone and bounded. Find the limit.

4. *(a)* Give an example of two divergent sequences such that their sum converges. 1

(b) Prove that the limit of a sequence of real numbers is unique. 2

(c) Prove that

$$\lim_{n\to\infty} x_n=0$$

if and only if

$$\lim_{n\to\infty}(|x_n|)=0$$ 3

(d) Establish the convergence or divergence of the following sequences (any *one*) : 4

(i) $x_n=\dfrac{(-1)^n n}{n+1}$

(ii) $x_n=\dfrac{n^2}{n+1}$

(iii) $x_n=\dfrac{2n^2+3}{n^2+1}$

(e) Define Cauchy sequence. Prove that a sequence of real numbers is Cauchy if and only if it is convergent. 1+4=5

Or

Establish the convergence or divergence of the sequence

$$y_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

for $n\in\mathbb{N}$. 5

(d) Show that

$$\lim_{n \to \infty} (b^n) = 0$$

if $0 < b < 1$. 4

Or

Show that

$$\lim_{n \to \infty} (c^{\frac{1}{n}}) = 1$$

for $c > 1$.

(e) State and prove the Monotone Convergence theorem. 5

Or

Let $Y := (y_n)$ be defined as $y_1 = 1$, $y_{n+1} = \frac{1}{4} y_n + 2$, $n \geq 1$. Show that (y_n) is monotone and bounded. Find the limit.

4\. (a) Give an example of two divergent sequences such that their sum converges. 1

(b) Prove that the limit of a sequence of real numbers is unique. 2

5. *(a)* State the Cauchy Criterion for convergence of a series. 1

(b) Prove that if

$$\sum_{n=1}^{\infty} x_n$$

converges then

$$\lim_{n \to \infty} (x_n) = 0$$ 3

(c) Prove that if

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. 3

(d) Show that the series

$$\sum_{n=1}^{\infty} x_n$$

converges if and only if the sequence $S = (s_k)$ of partial sums is bounded. 4

(e) Define absolute convergence. Show that if a series of real numbers is absolutely convergent then it is convergent. 1+3=4

(f) Let f be a positive, decreasing function on $\{t : t \geq 1\}$. Show that the series

$$\sum_{k=1}^{\infty} f(k)$$

converges if and only if the improper integral

$$\int_1^\infty f(t)dt = \lim_{b\to\infty}\int_1^b f(t)dt$$

exists. 5

Or

Show that the series

$$\sum_{n=1}^{\infty}\cos n$$

is divergent.

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